## Class Road Map - IT-I

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**Finals Week: December 9th–13th.**
Cumulative Outstanding Reading TODOs

- Chapters 1 and 2 in our book (Cover & Thomas, “Information Theory”) (including Fano’s inequality).
- Chapters 3 and 4 in our book (Cover & Thomas, “Information Theory”).
- Sections 11.1 through 11.3 in our book (Cover & Thomas, “Information Theory”).
- Chapter 4 in our book (Cover & Thomas, “Information Theory”).
- Chapter 5 in our book (Cover & Thomas, “Information Theory”).
- Chapter 13 in our book (Cover & Thomas, “Information Theory”) (there is no chapter on arithmetic coding but the lecture slides will be complete, or see MacKay’s online text).
- Chapter 7 in our book (Cover & Thomas, “Information Theory”) on channel capacity.
Homework

- Homework 1 on our assignment dropbox (https://canvas.uw.edu/courses/1319497/assignments), was due Tuesday, Oct 8th, 11:55pm.
- Homework 2 on our assignment dropbox (https://canvas.uw.edu/courses/1319497/assignments), due Friday 10/18/2019, 11:45pm.
- Homework 3 on our assignment dropbox (https://canvas.uw.edu/courses/1319497/assignments), due Tuesday 10/29/2019, 11:45pm.
- Homework 5 available soon.
Announcement: On Wed 11/13/2019

- This lecture will be given by our TA, Neeraja Abhyankar (but you probably already know that by now).
Place yourself back in the 1930s.

Analog communication model of the 1930s.

Q: Can we achieve perfect communication with an imperfect communication channel?

Q: Is there an upper bound on the information capable of being sent under different noise conditions?

If we increase the transmission rate over a noisy channel will the error rate increase?
Radio Communications

- Key: If we increase the transmission rate over a noisy channel will the error rate increase?
- Perhaps the only way to achieve error free communication is to have a rate of zero.
- The error profile we might expect to see is the following:

\[
P_e \rightarrow R
\]

- Here, probability of error \( P_e \) goes up linearly with the rate \( R \), with an intercept at zero.
- This was the prevailing wisdom at the time. Shannon was critical in changing that.
Simple Example

- Consider representing a signal by a sequence of numbers.
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We now know that any signal (either inherently discrete or continuous, under the right conditions) can be perfectly represented (or at least arbitrarily well) by a sequence of discrete numbers, and they can even be binary digits.
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- In such case, each number we repeat $k$ times, where $k$ is sufficiently large to ensure we can “decode” the original sequence with very small probability of error.
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- Rate of this code decreases but we can communicate reliably even if the channel is very noisy.
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- Rate of this code decreases but we can communicate reliably even if the channel is very noisy.
- Compare this idea to the figure on the following page.
Redundancy added to reduce errors

- Example: speaking numbers in AM radio, say “4,3,5,1,9” very likely could hear “4,static,5,1,static,” part of the message irreparably lost.
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- Can repeat $> 2 \times$ if we wish to decrease the chance of error.
- Also, don’t send the signal, send a representation (or a description, or encoding) of the signal to well fit the channel, and in order to be able to recover (decode) the signal.
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- Send information about a signal rather than the exact signal itself. Redundancy added for the sake of the channel, not for the sake of signal source or its contents. Also, rate need not go to zero as $P_e \to 0$! This was revolutionary in the 1930s/1940s!
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- Is Morse code an example of this from the 1800s?
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- Is Morse code an example of this from the 1800s? Not really, Morse is symbol code, directly for sending text, not for other types of signals. Moreover, no coding redundancy, the encoder (telegraph machine) had tonal redundancy but not to the degree done in 1940s.
A key idea

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  ![Diagram showing the relationship between source messages and received messages.](Diagram)

- This might restrict our possible set of source messages (in some cases severely, and thereby decrease our rate $R$), but if any message received in a region corresponds to only one source message, “perfect” communication can be achieved.
Discrete Channels

Definition 13.3.1 (discrete channel)

A discrete channel is one where there is an input alphabet $\mathcal{X}$, an output alphabet $\mathcal{Y}$, and a distribution $p(y|x)$ which is the probability of observing output $y$ after seeing $x$ as input.
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Definition 13.3.2 (memoryless channel)

A discrete channel is memoryless if \( y_t \), the output at time \( t \), is independent of all previous inputs, given \( x_t \). I.e., \( y_t \perp \perp x_{1:t-1} | x_t \).
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- We will see many instances of discrete memoryless channels (or just DMC).
- Recall back from lecture 1 our general model of communications:
Source message $W$, one of $M$ messages.
**Model of Communication**

- **Source** message $W$, one of $M$ messages.
- **Encoder** transforms this into a length-$n$ string of source symbols $X^n$ (we might call them “channel input symbols”).
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Note, $p(x, y) = p(x)p(y|z)$.

Noise $p(y|x)$
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- **Note**, $p(x, y) = p(x)p(y|x)$. $p(y|x)$ will model our channel and is fixed in most cases (we can’t control the channel). $p(x)$ is our source distribution, and we get to determine (and optimize over) it.
So we have a source $X$ governed by $p(x)$ and channel that transforms $X$ symbols to $Y$ symbols and which is governed by the conditional distribution $p(y|x)$.
Rates and Capacities

- So we have a source $X$ governed by $p(x)$ and channel that transforms $X$ symbols to $Y$ symbols and which is governed by the conditional distribution $p(y|x)$.
- These two items $p(x)$ and $p(y|x)$ is sufficient to compute the mutual information between $X$ and $Y$. 

\[
I(X;Y) = \sum_{x,y} p(x) p(y|x) \log \frac{p(y|x)}{\sum_x p(y|x')} p(x')
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$$I(X;Y) = I_{p(x)}(X,Y) = \sum_{x,y} p(x)p(y|x) \log \frac{p(y|x)}{p(y)}$$

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- We write this as $I(X;Y) = I_{p(x)}(X,Y)$, meaning implicitly the MI quantity is a function of the entire distribution $p(x)$, for a given fixed channel $p(y|x)$. Recall from L3, concave in $p(x)$ for fixed $p(y|x)$. 

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Prof. Jeff Bilmes
EE514a/Fall 2019/Info. Theory I – Lecture 13 - Nov 13th, 2019

L13 F13/32 (pg.39/147)
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We will often be optimizing over the input distribution $p(x)$ for a given fixed channel $p(y|x)$.
Rates and Capacities

**Definition 13.3.3 (information flow)**

The rate of information flow through a channel is given by $I(X; Y)$, the mutual information between $X$ and $Y$, in units of bits per channel use.
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Definition 13.3.4 (capacity)
The information capacity of a channel is the maximum information flow.

$$C \triangleq \max_{p(x) \in \Delta} I(X; Y)$$  \hspace{1cm} (13.3)

where $\Delta$ is the set of all possible probability distributions over source alphabet $\mathcal{X}$.
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Definition 13.3.5 (rate)
The rate $R$ of a code is measured in the number of bits per channel use.
For compression, if error exponent is positive, then error $\rightarrow 0$ exponentially fast as block length $\rightarrow \infty$. Note, $P_e \propto e^{-nE(R)}$. That is, $E(R) \rightarrow \sigma$ Error Exponent Only hope of reducing error was if $R > H$. Something “funny” happens at the entropy rate of the source distribution. Can't compress below this without incurring error.
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![Diagram](image)

- Error Exponent $E(R)$
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Radio Communications

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- Perhaps the only way to achieve error free communication is to have a rate of zero.
- The error profile we might expect to see is the following:

Here, probability of error $P_e$ goes up linearly with the rate $R$, with an intercept at zero.
- This was the prevailing wisdom at the time. Shannon was critical in changing that.
For communication, lower bound on probability of error becomes bounded away from 0 as the rate of the code $R$ goes above a fundamental quantity $C$. Note, $P_e \propto e^{-nE(R)}$. We will show: only way to get low error is $R < C$. Something funny happens at the point $C$, the channel capacity. Note that $C$ is not 0, so can still achieve “perfect” communication over a noisy channel as long as $R < C$. 
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We will do this. For now, think of \( C \) has being measured in units of “bits per channel use”
Examples of discrete memoryless channels (BSC)

- Noiseless binary channel, diagram shows $p(y|x)$

\[
X = \{0, 1\} \quad X \quad Y \quad Y = \{0, 1\}
\]

\[
\begin{array}{c}
0 \\ 1
\end{array} \quad \rightarrow \quad \begin{array}{c}
0 \\ 1
\end{array}
\]

So, $p(y = 0 | x = 0) = 1 = 1 - p(y = 1 | x = 0)$ and $p(y = 1 | x = 1) = 1 = 1 - p(y = 0 | x = 1)$, so channel is just an input copy.

One bit sent at a time is received without error, so capacity should be 1 bit (intuitively, we can reliably send one bit per channel usage).

\[
I(X;Y) = H(X) - H(X|Y) = H(X)
\]

In this case, so $C = \max_p p(x) I(X;Y) = \max_p p(x) H(X) = 1$.

Clearly, $p(0) = p(1) = \frac{1}{2}$ achieves this capacity.

Also, $p(0) = 1 = 1 - p(1)$ has $I(X;Y) = 0$, so achieves zero information flow.
Examples of discrete memoryless channels (BSC)

- Noiseless binary channel, diagram shows $p(y|x)$
  \[ X = \{0, 1\} \quad X \rightarrow 0 \]
  \[ Y = \{0, 1\} \quad Y \rightarrow 1 \]

- So, $p(y = 0|x = 0) = 1 = 1 - p(y = 1|x = 0)$ and $p(y = 1|x = 1) = 1 = 1 - p(y = 0|x = 1)$, so channel is just an input copy.
Examples of discrete memoryless channels (BSC)

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\[
\begin{align*}
\mathcal{X} &= \{0, 1\} & \mathcal{X} &\rightarrow 0 \\
0 &\rightarrow 0 & \mathcal{Y} &\rightarrow 1 \\
1 &\rightarrow 1 & \mathcal{Y} &= \{0, 1\}
\end{align*}
\]

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- Noiseless binary channel, diagram shows $p(y|x)$

  $\mathcal{X} = \{0, 1\}$  $\mathcal{Y} = \{0, 1\}$

  \[
  \begin{array}{c}
  0 \\
  1 \\
  \end{array}
  \quad \begin{array}{c}
  \rightarrow 0 \\
  \rightarrow 1 \\
  \end{array}
  \]

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  $Y = \{0, 1\}$

  $X \rightarrow 0$  
  $1 \rightarrow 1$

  $Y \rightarrow 0$  
  $1 \rightarrow 1$

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  0  0  1  1

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- Also, $p(0) = 1 = 1 - p(1)$ has $I(X; Y) = 0$, so achieves zero information flow.
Consider the channel

\[
\begin{align*}
X & \quad \xrightarrow{\frac{1}{2}} \quad 0 \\
0 & \quad \xleftarrow{\frac{1}{2}} \quad 1 \\
1 & \quad \xrightarrow{\frac{1}{2}} \quad 2 \\
1 & \quad \xleftarrow{\frac{1}{2}} \quad 3 \\
\end{align*}
\]

Here, \( p(Y = 0|X = 0) = p(Y = 1|X = 0) = 1/2 \) and \( p(Y = 2|X = 1) = p(Y = 3|X = 1) = 1/2 \).
Consider the channel

\[
\begin{array}{c}
0 \\
\frac{1}{2}
\end{array}
\quad \begin{array}{c}
\frac{1}{2}
\end{array}
\quad 1
\quad \begin{array}{c}
0
\end{array}
\quad \begin{array}{c}
\frac{1}{2}
\end{array}
\quad 2
\quad \begin{array}{c}
\frac{1}{2}
\end{array}
\quad 1
\quad \begin{array}{c}
\frac{1}{2}
\end{array}
\quad 3
\quad \begin{array}{c}
\frac{1}{2}
\end{array}
\quad Y
\end{array}
\]

- Here, \( p(Y = 0|X = 0) = p(Y = 1|X = 0) = 1/2 \) and \( p(Y = 2|X = 1) = p(Y = 3|X = 1) = 1/2 \).
- If we receive a 0 or 1, we know 0 was sent. If we receive a 2 or 3, a 1 was sent.
Consider the channel

\[ X \rightarrow Y \]

- Here, \( p(Y = 0|X = 0) = p(Y = 1|X = 0) = 1/2 \) and \( p(Y = 2|X = 1) = p(Y = 3|X = 1) = 1/2 \).
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- Thus, \( C = 1 \) since only two possible error free messages.
Noisy Channel with non-overlapping outputs

Consider the channel

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td></td>
<td>$\frac{1}{2}$</td>
<td></td>
</tr>
</tbody>
</table>

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- Same argument applies

$$I(X; Y) = H(X) - H(X|Y) = H(X).$$

$$\mathbb{E} DMC$$
Noisy Channel with non-overlapping outputs

Consider the channel

\[
\begin{array}{cccc}
0 & \xrightarrow{\frac{1}{2}} & 0 \\
0 & \xrightarrow{\frac{1}{2}} & 1 \\
1 & \xrightarrow{\frac{1}{2}} & 2 \\
1 & \xrightarrow{\frac{1}{2}} & 3 \\
\end{array}
\]

- Here, \( p(Y = 0|X = 0) = p(Y = 1|X = 0) = 1/2 \) and \( p(Y = 2|X = 1) = p(Y = 3|X = 1) = 1/2 \).
- If we receive a 0 or 1, we know 0 was sent. If we receive a 2 or 3, a 1 was sent.
- Thus, \( C = 1 \) since only two possible error free messages.
- Same argument applies \( I(X;Y) = H(X) - H(X|Y) = H(X) \).
- Again uniform distribution \( p(0) = p(1) = 1/2 \) achieves the capacity.
Permutation Channel

Consider the channel

\[
\begin{array}{c}
X \\
0 \\
1 \\
\end{array} \quad \begin{array}{c}
\rightarrow \ 0 \\
\rightarrow \ 1 \\
\end{array} \quad \begin{array}{c}
Y \\
0 \\
1 \\
\end{array}
\]

Here, \( p(Y = 1|X = 0) = p(Y = 0|X = 1) = 1 \).
Consider the channel

\[
\begin{array}{ccc}
X & 0 & Y \\
0 & \rightarrow & 0 \\
1 & \rightarrow & 1 \\
\end{array}
\]

- Here, \( p(Y = 1|X = 0) = p(Y = 0|X = 1) = 1 \).
- So output is a binary permutation (swap) of input.
Consider the channel \( X \rightarrow Y \):

- Here, \( p(Y = 1|X = 0) = p(Y = 0|X = 1) = 1 \).
- So output is a binary permutation (swap) of input.
- Thus, \( C = 1 \) no information lost.
Permutation Channel

Consider the channel

\[
\begin{array}{ccc}
X & \rightarrow & 0 \\
1 & \rightarrow & 1 \\
0 & \rightarrow & Y \\
1 & \rightarrow & 0
\end{array}
\]

- Here, \( p(Y = 1|X = 0) = p(Y = 0|X = 1) = 1 \).
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- In general, for alphabet of size \( k = |X| = |Y| \), let \( \sigma \) be a permutation, so that \( Y = \sigma(X) \).
Permutation Channel

Consider the channel

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\begin{array}{c}
 X & \rightarrow & Y \\
 0 & \rightarrow & 0 \\
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Thus, \( C = 1 \) no information lost.

In general, for alphabet of size \( k = |X| = |Y| \), let \( \sigma \) be a permutation, so that \( Y = \sigma(X) \).

Then \( C = \log k \).
Asside: on the optimization to compute the value $C$

- To maximize a given function $f(x)$, it is sufficient to show that $f(x) \leq \alpha$ for all $x$, and then find an $x^*$ such that $f(x^*) = \alpha$. 
Asside: on the optimization to compute the value $C$

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- We’ll be doing this over the next few slides when we want to compute $C = \max_{p(x)} I(X; Y)$ for fixed $p(y|x)$. 

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Prof. Jeff Bilmes
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- We’ll be doing this over the next few slides when we want to compute $C = \max_p I(X; Y)$ for fixed $p(y|x)$.
- The solution $p^*(x)$ that we find that achieves this maximum won’t necessarily be unique.
To maximize a given function \( f(x) \), it is sufficient to show that \( f(x) \leq \alpha \) for all \( x \), and then find an \( x^* \) such that \( f(x^*) = \alpha \).

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- The solution \( p^*(x) \) that we find that achieves this maximum won’t necessarily be unique.

- Also, the solution \( p^*(x) \) that we find won’t necessarily be the one that we end up, say, using when we wish to do channel coding.
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- Right now $C$ is just the result of a given optimization.
- We’ll see that $C$, as computed, is also the critical point for being able to channel code with vanishingly small error probability.
Asside: on the optimization to compute the value $C$

- To maximize a given function $f(x)$, it is sufficient to show that $f(x) \leq \alpha$ for all $x$, and then find an $x^*$ such that $f(x^*) = \alpha$.
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- Right now $C$ is just the result of a given optimization.
- We’ll see that $C$, as computed, is also the critical point for being able to channel code with vanishingly small error probability.
- The resulting $p^*(x)$ that we obtain as part of the optimization in order to compute $C$ won’t necessarily be the one that we use for actual coding (examples are forthcoming).
Noisy Typewriter

Consider the channel

A → [½ A, ½ B]
B → [½ B, ½ C]
C → [½ C, ½ D]
D → [½ D, ½ E]
E → [½ E, ½ F]
F → [½ F, ½ G]
G → [½ G, ½ H]
H → [½ H, ½ I]
I → [½ I, ½ J]
J → [½ J, ½ K]
K → [½ K, ½ L]
L → [½ L, ½ M]
M → [½ M, ½ N]
N → [½ N, ½ O]
O → [½ O, ½ P]
P → [½ P, ½ Q]
Q → [½ Q, ½ R]
R → [½ R, ½ S]
S → [½ S, ½ T]
T → [½ T, ½ U]
U → [½ U, ½ V]
V → [½ V, ½ W]
W → [½ W, ½ X]
X → [½ X, ½ Y]
Y → [½ Y, ½ Z]
Z → [½ Z, ½ A]

So 26 input symbols, and each symbol maps probabilistically to itself or its lexicographic neighbor. I.e., $p(A \rightarrow A) = p(A \rightarrow B) = \frac{1}{2}$, etc.

Each symbol always has chance of error, so how can we communicate without error? Choose subset of symbols that can be uniquely disambiguated on receiver side. Choose every other source symbol, A, C, E, ... Thus $A \rightarrow \{A, B\}$, $C \rightarrow \{C, D\}$, $E \rightarrow \{E, F\}$, etc. so that each received symbols has only one unique source symbol.

Capacity $C = \log_2 13$, visualized on left:

Q: what happens to $C$ when probabilities are not all $\frac{1}{2}$?
Consider the channel

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Noisy Typewriter

Consider the channel

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- Thus \( A \rightarrow \{A, B\} \), \( C \rightarrow \{C, D\} \), \( E \rightarrow \{E, F\} \), etc. so that each received symbol has only one unique source symbol.
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- Capacity \( C = \log_{13} \), visualized on left:
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Capacity \( C = \log 13 \), visualized on left:

Q: what happens to \( C \) when probabilities are not all 1/2?
Noisy Typewriter

- We can also compute the capacity more mathematically.
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\[
C = \max_{p(x)} I(X; Y) = \max_{p(x)} \left( H(Y) - H(Y|X) \right) \tag{13.4}
\]

\[
\forall x, \text{ when } X = x, \exists \text{ two equal prob. choices for } Y, \text{ hence } ...
\]

\[
= \max_{p(x)} H(Y) - 1 \tag{13.5}
\]

\[
= \log 26 - 1 = \log 13 \tag{13.6}
\]
Noisy Typewriter

- We can also compute the capacity more mathematically. E.g.,

\[
C = \max_{p(x)} I(X; Y) = \max_{p(x)} \left( H(Y) - H(Y | X) \right)
\]  

(13.4)

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\[
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\]  

(13.5)

\[
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\]  

(13.6)

- The \(\max_{p(x)} H(Y) = \log 26\) can be achieved by using the uniform distribution for \(p(x)\), for which when we choose any \(x\) symbol, there is equal likelihood of two \(Y\)'s being received.
We can also compute the capacity more mathematically. E.g.,

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The \( \max_{p(x)} H(Y) = \log 26 \) can be achieved by using the uniform distribution for \( p(x) \), for which when we choose any \( x \) symbol, there is equal likelihood of two \( Y \)s being received.

An alternatively \( p(x) \) would put zero probability on the alternates (B, D, F, etc.). In this case, we still have \( H(Y) = \log 26 \)
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- An alternatively \( p(x) \) would put zero probability on the alternates (B, D, F, etc.). In this case, we still have \( H(Y) = \log 26 \)

- So the capacity is the same in each case (\( \exists \text{ two } p(x) \text{ that achieved this} \) but only one is what we would use, say, for error free coding.)
Binary Symmetric Channel (BSC)

A bit that is sent will be flipped with probability $p$. $p(Y = 1 | X = 0) = p = 1 - p(Y = 0 | X = 0)$. $p(Y = 0 | X = 1) = p = p(Y = 1 | X = 1)$.

The BSC is an important channel since it is a simple model but at the same time captures some of the difficulties of more complicated channels.

Q: can we still achieve reliable (“guaranteed” error free) communication with this channel?

A: Yes, if $p < 1/2$ and if we do not ask for too high a transmission rate (which would be $R > C$), then we can. Actually, any $p \neq 1/2$ is sufficient.

Intuition: think about AEP and/or block coding.

But how to compute $C$, the capacity?
Binary Symmetric Channel (BSC)

A bit that is sent will be flipped with probability $p$. 

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Q: can we still achieve reliable (“guaranteed” error free) communication with this channel? A: Yes, if $p < 1/2$ and if we do not ask for too high a transmission rate (which would be $R > C$), then we can. Actually, any $p \neq 1/2$ is sufficient.
A bit that is sent will be flipped with probability $p$.

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Intuition: think about AEP and/or block coding.
Binary Symmetric Channel (BSC)

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But how to compute $C$ the capacity?
BSC Capacity

\[ I(X; Y) = H(Y) - H(Y|X) \]  

(13.8)
BSC Capacity

\[ I(X; Y) = H(Y) - H(Y|X) = H(Y) - \sum_x p(x) H(Y|X = x) \quad (13.7) \]

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BSC Capacity

\[
\begin{align*}
I(X; Y) &= H(Y) - H(Y|X) = H(Y) - \sum_x p(x)H(Y|X = x) \\
&= H(Y) - \sum_x p(x)H(p) = H(Y) - H(p)
\end{align*}
\]
BSC Capacity

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- To achieve the upper bound, need \( H(Y) = 1 \). Note that

\[ \Pr(Y = 1) \] (13.12)
BSC Capacity

\[
\begin{array}{c}
X \\
0 \\
p \\
1 \\
1-p \\
\end{array} \quad \begin{array}{c}
p \\
1-p \\
0 \\
1 \\
\end{array} \quad Y
\]

\[
I(X; Y) = H(Y) - H(Y|X) = H(Y) - \sum_x p(x) H(Y|X = x) \quad (13.7)
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\[
\Pr(Y = 1) = \Pr(Y = 1|X = 1)\Pr(X = 1) + \Pr(Y = 1|X = 0)\Pr(X = 0) \quad (13.9)
\]

\[
= p(1-p) + (1-p)p \quad (13.10)
\]

\[
H(Y) = 1 \quad (i.e., \Pr(X = 1) = 1/2) \quad (13.12)
\]
BSC Capacity

\[
I(X; Y) = H(Y) - H(Y|X) = H(Y) - \sum_x p(x)H(Y|X = x) \tag{13.7}
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\[
= (1 - p)\Pr(X = 1) + p(1 - \Pr(X = 1)) \tag{13.10}
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$$= p + (1 - 2p)\Pr(X = 1)$$
BSC Capacity

\[
\begin{array}{c}
\begin{array}{c}
\text{X} \\
\begin{array}{c}
0 \\
p \\
1 \\
1-p
\end{array}
\end{array}
\begin{array}{c}
1-p \\
p \\
p \\
1-p
\end{array}
\begin{array}{c}
\text{Y} \\
0 \\
1
\end{array}
\end{array}
\]

\[
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\]

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BSC Capacity

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- To achieve the upper bound, need \( H(Y) = 1 \). Note that

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\[ + \Pr(Y = 1|X = 0)\Pr(X = 0) \quad (13.10) \]

\[ = (1 - p)\Pr(X = 1) + p(1 - \Pr(X = 1)) \quad (13.11) \]

\[ = p + (1 - 2p)\Pr(X = 1) \quad (13.12) \]

- So \( H(Y) = 1 \) if \( H(X) = 1 \) (i.e., if \( \Pr(X = 1) = 1/2 \)).

- Thus, we get that \[ C = 1 - H(p) \] which happens when \( X \) is uniform.
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If $p \neq 1/2$, then we can communicate, albeit potentially slowly. E.g., if $p = 0.499$ then $C = 2.8854 \times 10^{-6}$ bits per channel use. So to send one bit, need to use the channel quite a bit.
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If \( p = 0 \) or \( p = 1 \), then \( C = 1 \) and we can get maximum possible rate (i.e., the capacity is one bit per channel use).
Decoding

- Lets temporarily look ahead to address this problem.
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- We can “decode” the source using the received string, source distribution, and the channel model $p(y|x)$ via Bayes rule. I.e.

$$
Pr(x|y) = \frac{\text{channel} \cdot \text{source}}{Pr(y)} = \frac{Pr(y|x)Pr(x)}{Pr(y)} = \frac{Pr(y|x)Pr(x)}{\sum_{x'} Pr(y|x')Pr(x')} \quad (13.13)
$$

If we get a particular $y$, we can compute $p(x|y)$ and make a decision based on that. I.e.,

$$
\hat{x} = \arg\max_x p(x|y)
$$

Error if $x \neq \hat{x}$, and $Pr(error) = Pr(x \neq \hat{x})$.

This is optimal decoding in that it minimizes the error, $Error(\bar{x}) = 1 - Pr(\bar{x}|y(x))$ when $y$ is received for sent $x$. This is minimal if we chose $\arg\max_x p(x|y)$ since the error becomes $1 - Pr(\hat{x}|y)$ which is minimal.
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Channel Capacity

Example DMC

Decoding

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Minimum Error Decoding

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- Many real world codes are such that computing the exact computation must be approximated (i.e., no known fast algorithm for minimum error or maximum likelihood decoding).
- Instead we do approximate inference algorithms (e.g., loopy belief propagation, message passing, etc.). These algorithms tend still to work very well in practice (achieve close to the capacity $C$).
- But before doing that, we need first to study more channels and the theoretical properties of the capacity $C$. 
The probability of dropping a bit is then \( \alpha \).

We want to compute capacity. Obviously, \( C = 1 \) if \( \alpha = 0 \).

\[
C = \max_{p(x)} I(X;Y) = \max_{p(x)} \left( H(Y) - H(Y|X) \right)
\] (13.14)

\[
= \max_{p(x)} H(Y) - H(\alpha)
\] (13.15)

So while \( H(Y) \leq \log 3 \), we want actual value of the capacity.

\( e \) is an erasure symbol, if that happens we don’t have access to the transmitted bit.
Binary Erasure Channel

- $e$ is an erasure symbol, if that happens we don’t have access to the transmitted bit.
- The probability of dropping a bit is then $\alpha$.

\[
\begin{align*}
X & \quad 0 \quad \rightarrow \quad 0 \\
0 & \quad \alpha \quad \rightarrow \quad e \\
1 & \quad \alpha \quad \rightarrow \quad 1 \\
1 \quad 1 - \alpha \quad \rightarrow \quad 1
\end{align*}
\]
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$$C = \max_{p(x)} I(X; Y)$$

(13.15)
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$$C = \max_{p(x)} I(X; Y) = \max_{p(x)} (H(Y) - H(Y|X))$$  \hspace{1cm} (13.14)

$$H(Y) = \sum_y p(y) \log \frac{1}{p(y)}$$  \hspace{1cm} (13.15)
**Binary Erasure Channel**

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$$H(Y)$$
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\[ H(Y) = H(Y, E) \]
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- Let \( E = \{Y = e\} \). Then

\[
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Binary Erasure Channel

- let $E = \{Y = e\}$. Then
  
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- Let $\pi = \Pr(X = 1)$. 

\[ 
\begin{array}{ccc}
0 & \xrightarrow{1 - \alpha} & 0 \\
\alpha & \xrightarrow{\alpha} & e \\
1 & \xrightarrow{1 - \alpha} & 1 \\
\end{array}
\]
Binary Erasure Channel

- Let $E = \{Y = e\}$. Then
  \[
  H(Y) = H(Y, E) = H(E) + H(Y | E)
  \]

- Let $\pi = \Pr(X = 1)$. Then
  \[
  H(Y) = H((1 - \pi)(1 - \alpha), \alpha, \pi(1 - \alpha)) \quad \text{(13.16)}
  \]
  \[
  = \alpha H(Y | e) + (1 - \alpha) H(Y | \neg e) \quad \text{(13.17)}
  \]
Binary Erasure Channel

- Let $E = \{Y = e\}$. Then

  $$H(Y) = H(Y, E) = H(E) + H(Y|E)$$

- Let $\pi = \Pr(X = 1)$. Then

  $$H(Y) = \begin{cases} 
  H((1 - \pi)(1 - \alpha)), & \text{if } Y=0 \\
  \alpha, & \text{if } Y=e \\
  \pi(1 - \alpha), & \text{if } Y=1 
  \end{cases}
  \quad (13.16)$$

  $$= H(\alpha) + (1 - \alpha)H(\pi) \quad (13.17)$$
**Binary Erasure Channel**

1. Let \( E = \{Y = e\} \). Then
   \[
   H(Y) = H(Y, E) = H(E) + H(Y|E)
   \]

2. Let \( \pi = \Pr(X = 1) \). Then
   \[
   H(Y) = H((1 - \pi)(1 - \alpha), \alpha, \pi(1 - \alpha))
   \]
   \[
   = H(\alpha) + (1 - \alpha)H(\pi)
   \]

3. This last equality follows since \( H(E) = H(\alpha) \), and
   \[
   H(Y|E) = \alpha H(Y|Y = e) + (1 - \alpha)H(Y|Y \neq e) = \alpha \cdot 0 + (1 - \alpha)H(\pi)
   \]
Then we get

\[ C = \max_{p(x)} H(Y) - H(\alpha) \]  

\[ = \max_{\pi} \left( (1 - \alpha)H(\pi) + H(\alpha) \right) - H(\alpha) \]  

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Best capacity when \( \pi = 1/2 = \Pr(X = 1) = \Pr(X = 0) \).

This makes sense, loose \( \alpha \)% of the bits of original capacity.
Then we get

\[ C = \max_{p(x)} H(Y) - H(\alpha) \quad (13.18) \]

\[ = \max_\pi \left( (1 - \alpha)H(\pi) + H(\alpha) \right) - H(\alpha) \quad (13.19) \]

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