EE514a – Information Theory I
Fall Quarter 2019

Prof. Jeff Bilmes

University of Washington, Seattle
Department of Electrical & Computer Engineering
Fall Quarter, 2019
https://class.ece.uw.edu/514/bilmes/ee514_fall_2019/

Lecture 14 - Nov 18th, 2019
Class Road Map - IT-I

- L1 (9/25): Overview, Communications, Information, Entropy
- L2 (9/30): Entropy, Mutual Information, KL-Divergence
- L3 (10/2): More KL, Jensen, more Venn, Log Sum, Data Proc. Inequality
- L5 (10/9): M. of Conv, AEP
- L6 (10/14): AEP, Source Coding, Types
- LX (10/16): Makeup
- L7 (10/21): Types, Univ. Src Coding, Stoc. Procs, Entropy Rates
- L8 (10/23): Entropy rates, HMMs, Coding
- L9 (10/28): Kraft ineq., Shannon Codes, Kraft ineq. II, Huffman
- L10 (10/30): Huffman, Shannon/Fano/Elias
- L11 (11/4): Shannon/Fano/Elias, Games
- LXX (11/6): In class midterm exam
- L12 (11/11): Vet’s day, makeup lecture: Arith. Coding, Background On Channel Capacity
- L13 (11/13): Channel Capacity, Ex. DMC
- L15 (11/20):
- L16 (11/25):
- L17 (11/27):
- L18 (12/2):
- L19 (12/4):
- LXX (12/10): Final exam

Finals Week: December 9th–13th.
Cumulative Outstanding Reading TODOs

- Chapters 1 and 2 in our book (Cover & Thomas, “Information Theory”) (including Fano’s inequality).
- Chapters 3 and 4 in our book (Cover & Thomas, “Information Theory”).
- Sections 11.1 through 11.3 in our book (Cover & Thomas, “Information Theory”).
- Chapter 4 in our book (Cover & Thomas, “Information Theory”).
- Chapter 5 in our book (Cover & Thomas, “Information Theory”).
- Chapter 13 in our book (Cover & Thomas, “Information Theory”) (there is no chapter on arithmetic coding but the lecture slides will be complete, or see MacKay’s online text).
- Chapter 7 in our book (Cover & Thomas, “Information Theory”) on channel capacity.
Homework

- Homework 1 on our assignment dropbox (https://canvas.uw.edu/courses/1319497/assignments), was due Tuesday, Oct 8th, 11:55pm.

- Homework 2 on our assignment dropbox (https://canvas.uw.edu/courses/1319497/assignments), due Friday 10/18/2019, 11:45pm.

- Homework 3 on our assignment dropbox (https://canvas.uw.edu/courses/1319497/assignments), due Tuesday 10/29/2019, 11:45pm.


- Homework 5 available very soon.
Discrete Channels

**Definition 14.2.1 (discrete channel)**

A discrete channel is one where there is an input alphabet $\mathcal{X}$, an output alphabet $\mathcal{Y}$, and a distribution $p(y|x)$ which is the probability of observing output $y$ after seeing $x$ as input.

**Definition 14.2.2 (memoryless channel)**

A discrete channel is memoryless if $y_t$, the output at time $t$, is independent of all previous inputs, given $x_t$. I.e., $y_t \perp x_{1:t-1} | x_t$.

- We will see many instances of discrete memoryless channels (or just DMC).
- Recall back from lecture 1 our general model of communications:
Model of Communication

\[ \text{noise } p(y|x) \]

\[ \begin{align*}
\text{source} & \xrightarrow{} \text{encoder} & \text{channel} & \xrightarrow{} \text{decoder} & \text{receiver} \\
W & \rightarrow X^n & Y^n & \rightarrow \hat{W} \\
\text{n = } \log M \text{ bits} & \text{n log } |X| \text{ bits} & \text{n log } |Y| \text{ bits} 
\end{align*} \]

- **Source** message \( W \), one of \( M \) messages.
- **Encoder** transforms this into a length-\( n \) string of source symbols \( X^n \) (we might call them “channel input symbols”).
- **Noisy channel** distorts this message into a length-\( n \) string of receiver symbols \( Y^n \) (we might call them “channel output symbols”).
- **Decoder** attempts to reconstruct original message as best as possible, comes up with \( \hat{W} \), one of \( M \) possible sent messages.
- **Note**, \( p(x, y) = p(x)p(y|x) \). \( p(y|x) \) will model our channel and is fixed in most cases (we can’t control the channel). \( p(x) \) is our source distribution, and we get to determine (and optimize over) it.
Rates and Capacities

**Definition 14.2.1 (information flow)**

The rate of information flow through a channel is given by \( I(X; Y) \), the mutual information between \( X \) and \( Y \), in units of bits per channel use.

**Definition 14.2.2 (capacity)**

The information capacity of a channel is the maximum information flow.

\[
C \triangleq \max_{p(x) \in \Delta} I(X; Y) \tag{14.3}
\]

where \( \Delta \) is the set of all possible probability distributions over source alphabet \( \mathcal{X} \). Thus, \( C \) is the maximum number of bits sent over the channel per channel use.

**Definition 14.2.3 (rate)**

The rate \( R \) of a code is measured in the number of bits per channel use.
For communication, lower bound on probability of error becomes bounded away from 0 as the rate of the code $R$ goes above a fundamental quantity $C$. Note, $P_e \propto e^{-nE(R)}$.

That is, we have a “dual” situation to entropy compression, i.e.,

We will show: only way to get low error is $R < C$. Something funny happens at the point $C$, the channel capacity.

Note that $C$ is not 0, so can still achieve “perfect” communication over a noisy channel as long as $R < C$. 
For the moment, all we known about \( C \triangleq \max_{p(x) \in \Delta} I(X; Y) \) is its definition, which is the result of an optimization problem.

We will do this. For now, think of \( C \) has being measured in units of “bits per channel use”.

not yet connected it to communications, and to communicating about or below the rate \( C \).
A key idea

- If we choose the messages carefully at the sender, then with very high probability, they will be uniquely identifiable at the receiver.

- The idea is that we choose the source messages that (tend to) not have any ambiguity (or have any overlap) at the receiver end. i.e.,

This might restrict our possible set of source messages (in some cases severely, and thereby decrease our rate $R$), but if any message received in a region corresponds to only one source message, “perfect” communication can be achieved.
Noisy Typewriter

Consider the channel

- So 26 input symbols, and each symbol maps probabilistically to itself or its lexicographic neighbor.
- I.e., \( p(A \rightarrow A) = p(A \rightarrow B) = \frac{1}{2}, \) etc.
- Each symbol always has chance of error, so how can we communicate without error?
- Choose subset of symbols that can be uniquely disambiguated on receiver side. Choose every other source symbol, A, C, E, . . .
- Thus \( A \rightarrow \{A, B\}, \) \( C \rightarrow \{C, D\}, \) \( E \rightarrow \{E, F\}, \) etc. so that each received symbols has only one unique source symbol.
- Capacity \( C = \log 13, \) visualized on left:

Q: what happens to \( C \) when probabilities are not all \( 1/2? \)
BSC Capacity

\[ I(X; Y) = H(Y) - H(Y | X) = H(Y) - \sum_x p(x)H(Y | X = x) \]  \hspace{1cm} (14.6)

\[ = H(Y) - \sum_x p(x)H(p) = H(Y) - H(p) \leq 1 - H(p) \]  \hspace{1cm} (14.7)

- To achieve the upper bound, need \( H(Y) = 1 \). Note that

\[ \Pr(Y = 1) = \Pr(Y = 1 | X = 1)\Pr(X = 1) \]  \hspace{1cm} (14.8)

\[ + \Pr(Y = 1 | X = 0)\Pr(X = 0) \]  \hspace{1cm} (14.9)

\[ = (1-p)\Pr(X = 1) + p(1 - \Pr(X = 1)) \]  \hspace{1cm} (14.10)

\[ = p + (1 - 2p)\Pr(X = 1) \]  \hspace{1cm} (14.11)

- So \( H(Y) = 1 \) if \( H(X) = 1 \) (i.e., if \( \Pr(X = 1) = 1/2 \)).

- Thus, we get that \( C = 1 - H(p) \) which happens when \( X \) is uniform.
Ternary Confusion Channel

\[ P(Y = j | X = ?) = \frac{1}{2}. \]
Ternary Confusion Channel

- $P(Y = j | X = ?) = 1/2$.
- Whenever the symbol “?” is input, the output is random.
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- Other inputs are reliable.
Ternary Confusion Channel

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- Whenever the symbol “?” is input, the output is random.
- Other inputs are reliable.
- Thus, $C = 1$ bit.
**Definition 14.4.1**

A channel is **symmetric** if rows of the channel transmission matrix \( p(y|x) \) are permutations of each other, and columns of this matrix are permutations of each other.
Symmetric Channels

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![BSC](chart.png)
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Theorem 14.4.2
For weakly symmetric channels, we have that

$$C = \log |\mathcal{Y}| - H(r)$$  \hspace{1cm} (14.1)

where $r$ is the row of the transmission matrix.
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For weakly symmetric channels, we have that

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where $r$ is the row of the transmission matrix.

This follows immediately since

$$I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(r) \leq \log |\mathcal{Y}| - H(r)$$
Properties of (Information) Channel Capacity $C$

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- $C \leq \log |\mathcal{X}|$ since $C = \max_p(I(X;Y)) \leq \max H(X) = \log |\mathcal{X}|$.
- $C \leq \log |\mathcal{Y}|$ for same reason. Thus, the alphabet sizes limit the transmission rate.
- $I(X;Y) = I_{p(x)}(X;Y)$ is a continuous function of $p(x)$. 

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L14 F15/51 (pg.25/202)
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- $C \leq \log |\mathcal{Y}|$ for same reason. Thus, the alphabet sizes limit the transmission rate.
- $I(X; Y) = I_{p(x)}(X; Y)$ is a continuous function of $p(x)$.
- Recall, $I(X; Y)$ is a concave function of $p(x)$ for fixed $p(y|x)$. 

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- $C \leq \log |\mathcal{Y}|$ for same reason. Thus, the alphabet sizes limit the transmission rate.
- $I(X;Y) = I_{p(x)}(X;Y)$ is a continuous function of $p(x)$.
- Recall, $I(X;Y)$ is a concave function of $p(x)$ for fixed $p(y|x)$.
  Thus, $I_{\lambda p_1 + (1-\lambda)p_2}(X;Y) \geq \lambda I_{p_1}(X;Y) + (1-\lambda)I_{p_2}(X;Y)$. 

Interestingly, since concave, this makes computing something like the capacity easier. I.e., a local maximum is a global maximum, and computing the capacity for a general channel model is a convex optimization procedure.
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$I(X; Y) = I_{p(x)}(X; Y)$ is a continuous function of $p(x)$.

Recall, $I(X; Y)$ is a concave function of $p(x)$ for fixed $p(y|x)$. Thus, $I_{\lambda p_1 + (1-\lambda)p_2}(X; Y) \geq \lambda I_{p_1}(X; Y) + (1 - \lambda)I_{p_2}(X; Y)$.

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- $C \leq \log |\mathcal{Y}|$ for same reason. Thus, the alphabet sizes limit the transmission rate.
- $I(X; Y) = I_{p(x)}(X; Y)$ is a continuous function of $p(x)$.
- Recall, $I(X; Y)$ is a concave function of $p(x)$ for fixed $p(y|x)$. Thus, $I_{\lambda p_1 + (1-\lambda)p_2}(X; Y) \geq \lambda I_{p_1}(X; Y) + (1-\lambda)I_{p_2}(X; Y)$.
- Interestingly, since concave, this makes computing something like the capacity easier. I.e., a local maximum is a global maximum, and computing the capacity for a general channel model is a convex optimization procedure.
- Recall also, $I(X; Y)$ is a convex function of $p(y|x)$ for fixed $p(x)$. 
Shannon’s 2nd Theorem

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\( C \) is the maximum number of bits (on average, per channel use) that we can transmit over a channel reliably.
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$C$ is the maximum number of bits (on average, per channel use) that we can transmit over a channel reliably.

- Here, “reliably” means with vanishingly small and exponentially decreasing probability of error as the block length gets longer. We can easily make this probability essentially zero.
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- Here, “reliably” means with vanishingly small and exponentially decreasing probability of error as the block length gets longer. We can easily make this probability essentially zero.
- Conversely, if we try to push \( > C \) bits through the channel, error quickly goes to 1.
Intuition of this we’ve already seen in the noisy typewriter and the region partitioning.
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Slightly more precisely, this is a sort of bin packing problem.

We’ve got a region of possible codewords, and we pack as many smaller non-overlapping bins into the region as possible.

The smaller bins correspond to the noise in the channel, and the packing problem depends on the underlying “shape”
Shannon’s 2nd Theorem: Bin packing intuition

- Bin packing: not really a partition, since there might be wasted space, also depending on the bin and region shapes.
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The same thing is true for conditional entropy.
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- The same thing is true for conditional entropy.
- That is, for a typical input $X$, there are $\approx 2^{nH(Y|X)}$ output sequences.
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That is, for a typical input $X$, there are $\approx 2^{nH(Y|X)}$ output sequences.

Overall, there are $2^{nH(Y)}$ typical output sequences, and we know that $2^{nH(Y)} \geq 2^{nH(Y|X)}$. 
Shannon’s 2nd Theorem: Intuition

- Goal: find a non-confusable subset of the inputs that produce disjoint output sequences (as in picture).
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\text{sequences, but the best we can do (in terms of maximizing the number of non-confusable inputs) is when there is no (high probability) overlap on the output. This is assumed in the above.}
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- So the number of non-confusable inputs is

$$\leq \frac{2^{nH(Y)}}{2^{nH(Y|X)}} = 2^{n(H(Y) - H(Y|X))} = 2^{nI(X;Y)} \quad (14.2)$$
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  \]
- In non-ideal case, there could be overlap of the typical $Y$-given-$X$ sequences, but the best we can do (in terms of maximizing the number of non-confusable inputs) is when there is no (high probability) overlap on the output. This is assumed in the above.
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\]  \hspace{1cm} (14.3)

- We can view this as a volume. \(2^n H(Y)\) is the total number of possible slots, while \(2^n H(Y | X)\) is the number of slots taken up (on average) for a given source word. Thus, the number of source words that can be used is the ratio.
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- Now of course, to maximize this number, for a fixed channel \( p(y|x) \), we find the best \( p(x) \) which gives \( I(X;Y) = C \), which is the log of the maximum number of inputs possible to use.
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- This is the capacity.
Some Definitions

- **Reminder**: model of communication:

  \[ p(y|x) \]

  \[
  W = \log M \text{ bits} \\
  X^n = n \log |\mathcal{X}| \text{ bits} \\
  Y^n = n \log |\mathcal{Y}| \text{ bits} \\
  \hat{W} = \text{decoding via guess} \]

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**Ex. DMC**

**Properties**

**Joint AEP**

**Shannon’s 2nd Theorem**
Some Definitions

- Reminder: model of communication:
  - **noise** $p(y|x)$

  ![Diagram of communication model]

  - $W$: $n = \log M$ bits
  - $X^n$: $n \log |\mathcal{X}|$ bits
  - $Y^n$: $n \log |\mathcal{Y}|$ bits
  - $\hat{W}$

- Message $W \in \{1, \ldots, M\}$ requiring $\log M$ bits per message.
Some Definitions

- Reminder: model of communication: 
  \[ \text{noise } p(y|x) \]

- Message \( W \in \{1, \ldots, M\} \) requiring \( \log M \) bits per message.
- Signal sent through channel \( X^n(W) \), a random codeword.
Some Definitions

- Reminder: model of communication: \( \text{noise } p(y|x) \)

\[
\begin{align*}
\text{source} & \quad \text{encoder} & \quad \text{channel} & \quad \text{decoder} & \quad \text{receiver} \\
W & \quad X^n & \quad Y^n & \quad \hat{W} \\
n = \log M \text{ bits} & \quad n \log |\mathcal{X}| \text{ bits} & \quad n \log |\mathcal{Y}| \text{ bits} \\
\end{align*}
\]

- Message \( W \in \{1, \ldots, M\} \) requiring \( \log M \) bits per message.
- Signal sent through channel \( X^n(W) \), a random codeword.
- Received signal from channel \( Y^n \sim p(y^n|x^n) \)
Some Definitions

- Reminder: model of communication: **noise** $p(y|x)$

![Diagram of communication model](source encoder channel decoder receiver)

- Message $W \in \{1, \ldots, M\}$ requiring $\log M$ bits per message.
- Signal sent through channel $X^n(W)$, a random codeword.
- Received signal from channel $Y^n \sim p(y^n|x^n)$
- Decoding via guess $\hat{W} = g(Y^n)$. 

\[ \hat{W} \sim \prod_i p(y_i: 1s) \]
Some Definitions

- Reminder: model of communication:
  - **noise** \( p(y|x) \)

  ![Communication Model Diagram]

  - Message \( W \in \{1, \ldots, M\} \) requiring \( \log M \) bits per message.
  - Signal sent through channel \( X^n(W) \), a random codeword.
  - Received signal from channel \( Y^n \sim p(y^n|x^n) \)
  - Decoding via guess \( \hat{W} = g(Y^n) \).
  - Discrete memoryless channel (DMC) \( (X, p(y|x), Y) \)
Some Definitions

- Reminder: model of communication:
  \[ \text{noise} \ p(y|x) \]

\[ \begin{align*}
\text{source} & \rightarrow \text{encoder} & \rightarrow \text{channel} & \rightarrow \text{decoder} & \rightarrow \text{receiver} \\
W & & X^n & & Y^n & & \hat{W} \\
\text{n = log } M \text{ bits} & \quad \text{n log } |\mathcal{X}| \text{ bits} & \quad \text{n log } |\mathcal{Y}| \text{ bits} \\
\end{align*} \]

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- \( n^{\text{th}} \) extension to channel is \( (\mathcal{X}^n, p(y^n|x^n), \mathcal{Y}^n) \)
Some Definitions

- **Reminder**: model of communication:
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- **Signal** sent through channel \( X^n(W) \), a random codeword.
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- **Feedback** if \( x_k \) can use both previous inputs and outputs.
Some Definitions

- Reminder: model of communication: noise $p(y|x)$
- Message $W \in \{1, \ldots, M\}$ requiring $\log M$ bits per message.
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- Discrete memoryless channel (DMC) $(\mathcal{X}, p(y|x), \mathcal{Y})$
- $n^{th}$ extension to channel is $(\mathcal{X}^n, p(y^n|x^n), \mathcal{Y}^n)$
- Feedback if $x_k$ can use both previous inputs and outputs.
- No feedback if $p(x_k|x_{1:k-1}, y_{1:k-1}) = p(x_k|x_{1:k-1})$. We’ll analyze feedback a bit later.
Definition 14.4.4 \((M, n)\) code

An \((M, n)\) code for channel \((\mathcal{X}, p(y|x), \mathcal{Y})\) is:

1. An index set \(\{1, 2, \ldots, M\}\)
2. An encoding function \(X^n : \{1, 2, \ldots, M\} \rightarrow \mathcal{X}^n\) yielding codewords \(X^n(1), X^n(2), X^n(3), \ldots, X^n(M)\).
3. A decoding function, i.e., \(g : \mathcal{Y}^n \rightarrow \{1, 2, \ldots, M\}\) which makes a "guess" about the original message given the channel output.

In an \((M, n)\) code, \(M\) is the number of possible messages to be sent, and \(n\) is the number of channel uses by the codewords of the code.
Definition 14.4.4 ((M, n) code)

An (M, n) code for channel (X, p(y|x), Y) is:

1. An index set \( \{1, 2, \ldots, M\} \)
**Definition 14.4.4 ((M, n) code)**

An \((M, n)\) code for channel \((\mathcal{X}, p(y|x), \mathcal{Y})\) is:

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\[X^n(w) = \text{the length-}n \text{ codeword for message } w \in \{1, 2, \ldots, M\}.\]
(M, n) code

Definition 14.4.4 ((M, n) code)

An (M, n) code for channel (X, p(y|x), Y) is:

1. An index set \( \{1, 2, \ldots, M\} \)

2. An encoding function \( X^n : \{1, 2, \ldots, M\} \rightarrow X^n \) yielding codewords \( X^n(1), X^n(2), X^n(3), \ldots, X^n(M) \). Each source message has a codeword, and each codeword is \( n \) code symbols.

\[ \text{code symbols come from the} \ \mathcal{X} \ \text{alphabet.} \]
Definition 14.4.4 ((M, n) code)

An (M, n) code for channel (X, p(y|x), Y) is:

1. An index set \{1, 2, \ldots, M\}

2. An encoding function \(X^n : \{1, 2, \ldots, M\} \rightarrow X^n\) yielding codewords \(X^n(1), X^n(2), X^n(3), \ldots, X^n(M)\). Each source message has a codeword, and each codeword is \(n\) code symbols.

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- In an \((M, n)\) code, \(M = \) the number of possible messages to be sent, and \(n = \) number of channel uses by the codewords of the code.
Definition of Error

**Definition 14.4.5** (Probability of Error $\lambda_i$ for message $i \in \{1, \ldots, M\}$)

$$
\lambda_i \triangleq \Pr(g(Y^n) \neq i | X^n = X^n(i)) = \sum_{y^n \in \mathcal{Y}^n} p(y^n | X^n(i)) 1(g(y^n) \neq i)
$$

$$
\rho_c(g(y^n) \neq i | X^n = x^n(i))
$$

(14.5)
Definition of Error

**Definition 14.4.5 (Probability of Error $\lambda_i$ for message $i \in \{1, \ldots, M\}$)**

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\lambda_i \triangleq \Pr(g(Y^n) \neq i | X^n = X^n(i)) = \sum_{y^n \in \mathcal{Y}^n} p(y^n | X^n(i)) 1(g(y^n) \neq i)
$$

(14.5)

**Definition 14.4.6 (Max probability of Error $\lambda^{(n)}$ for $(M, n)$ code)**

$$
\lambda^{(n)} \triangleq \max_{i \in \{1, 2, \ldots, M\}} \lambda_i
$$

(14.6)
Definition 14.4.7 (Average probability of error $P_e^{(n)}$)

\[
P_e^{(n)} = \frac{1}{M} \sum_{i=1}^{M} \lambda_i = \Pr(I \neq g(Y^n))
\]

(14.7)

where $I$ is a r.v. with probability $\Pr(I = i)$ according to a uniform source distribution . . .

\[
= E(1(I \neq g(Y^n))) = \sum_{i=1}^{M} \Pr(g(Y^n) \neq i | X^n = X^n(i))p(i)
\]

(14.8)

with $p(i) = 1/M$. 
Definition 14.4.7 (Average probability of error $P_{e}^{(n)}$)

$$P_{e}^{(n)} = \frac{1}{M} \sum_{i=1}^{M} \lambda_{i} = \Pr(I \neq g(Y^{n}))$$  \hspace{1cm} (14.7)

where $I$ is a r.v. with probability $\Pr(I = i)$ according to a uniform source distribution . . .

$$= E(1(I \neq g(Y^{n}))) = \sum_{i=1}^{M} \Pr(g(Y^{n}) \neq i|X^{n} = X^{n}(i))p(i)$$  \hspace{1cm} (14.8)

with $p(i) = 1/M$.

- A key Shannon’s result is that a small average probability of error means we must have a small maximum probability of error!
Definition 14.4.8 (Rate $R$ of an $(M, n)$ code)

$$ R = \frac{\log M}{n} = \frac{\text{total num. of bits in a source message}}{\text{total num. of channel uses needed to send a message}} $$ (14.9)
Definition 14.4.8 (Rate $R$ of an $(M, n)$ code)

$$R = \frac{\log M}{n} = \frac{\text{total num. of bits in a source message}}{\text{total num. of channel uses needed to send a message}}$$

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- The rate $R$ is in units of bits per channel use, or bits per transmission.
Definition 14.4.8 (Rate $R$ of an $(M, n)$ code)

\[ R = \frac{\log M}{n} = \frac{\text{total num. of bits in a source message}}{\text{total num. of channel uses needed to send a message}} \]

(14.9)

- The rate $R$ is in units of bits per channel use, or bits per transmission.

Definition 14.4.9 (Achievability for a given channel)

A given rate $R$ is achievable for a given channel if \( \exists \) a sequence of \( \lceil 2^{nR} \rceil, n \) codes such that the maximal probability of error \( \lambda^{(n)} \rightarrow 0 \) as \( n \rightarrow \infty \).
Definition 14.4.10 (Capacity of a DMC)

The capacity of a DMC is the largest possible achievable rate.

- So the capacity of a DMC is the rate beyond which the error won’t any longer go to zero with increasing $n$. 
Capacity

**Definition 14.4.10 (Capacity of a DMC)**

The capacity of a DMC is the largest possible achievable rate.

- So the capacity of a DMC is the rate beyond which the error won't any longer go to zero with increasing $n$.
- **Note:** this is a different notion of capacity that we encountered before.
Definition 14.4.10 (Capacity of a DMC)

The capacity of a DMC is the largest possible achievable rate.

- So the capacity of a DMC is the rate beyond which the error won’t any longer go to zero with increasing $n$.
- Note: this is a different notion of capacity that we encountered before.
- Before we defined $C = \max_{p(x)} I(X; Y)$ as the “information capacity”
Definition 14.4.10 (Capacity of a DMC)

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- Here we are defining something called the “capacity of a DMC”.
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- Before we defined $C = \max_{p(x)} I(X; Y)$ as the “information capacity”
- Here we are defining something called the “capacity of a DMC”.
- We have not yet compared the two
Definition 14.4.10 (Capacity of a DMC)
The capacity of a DMC is the largest possible achievable rate.

- So the capacity of a DMC is the rate beyond which the error won’t any longer go to zero with increasing $n$.
- Note: this is a different notion of capacity that we encountered before.
- Before we defined $C = \max_p(x) I(X; Y)$ as the “information capacity”
- Here we are defining something called the “capacity of a DMC”.
- We have not yet compared the two (but of course we will 😊).
**Definition 14.4.11 (Joint typicality of a set of sequences)**

A set of sequences \( \{(x_1:n, y_1:n)\} \) w.r.t. \( p(x, y) \) is jointly typical \( (\in A_{\epsilon}^{(n)}) \) as per the following definition:

\[
A_{\epsilon}^{(n)} = \left\{ (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n : \right. \\
\left. \sum_{(x_i, y_i)} p(x_i, y_i) = \prod_{i=1}^{n} p(x_i, y_i) \leq \epsilon \right\}
\] (14.10)

\[
\text{with } p(x^n, y^n) = \prod_{i=1}^{n} p(x_i, y_i).
\] (14.13)
Definition 14.4.11 (Joint typicality of a set of sequences)

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\[
A_{\epsilon}^{(n)} = \left\{ (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n : \right. \\
\left. \begin{array}{l}
a) \quad \left| -\frac{1}{n} \log p(x^n) - H(X) \right| < \epsilon, \quad x\text{-typical} \\
\end{array} \right. \tag{14.11}
\]

with \( p(x^n, y^n) = \prod_{i=1}^{n} p(x_i, y_i) \).
Definition 14.4.11 (Joint typicality of a set of sequences)

A set of sequences $\{(x_{1:n}, y_{1:n})\}$ w.r.t. $p(x, y)$ is jointly typical ($\in A_{\epsilon}^{(n)}$) as per the following definition:

$$A_{\epsilon}^{(n)} = \left\{ (x^n, y^n) \in X^n \times Y^n : \right\}$$

\begin{align}
\text{a)} & \quad \left| -\frac{1}{n} \log p(x^n) - H(X) \right| < \epsilon, \quad x\text{-typical} \\
\text{b)} & \quad \left| -\frac{1}{n} \log p(y^n) - H(Y) \right| < \epsilon, \quad y\text{-typical}
\end{align}

\begin{equation}
\right\} \quad \text{with } p(x^n, y^n) = \prod_{i=1}^{n} p(x_i, y_i).
\end{equation}
Definition 14.4.11 (Joint typicality of a set of sequences)

A set of sequences \( \{(x_1^n, y_1^n)\} \) w.r.t. \( p(x, y) \) is jointly typical (\( \in A^{(n)}_\epsilon \)) as per the following definition:

\[
A^{(n)}_\epsilon = \left\{ (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n : \right. \\
\left. \begin{array}{l}
\frac{1}{n} \log p(x^n) - H(X) < \epsilon, \quad \text{x-typical} \\
\frac{1}{n} \log p(y^n) - H(Y) < \epsilon, \quad \text{y-typical} \\
\frac{1}{n} \log p(x^n, y^n) - H(X, Y) < \epsilon, \quad (x, y)\text{-typical}
\end{array} \right. \\
\right\} \tag{14.13}
\]

with \( p(x^n, y^n) = \prod_{i=1}^{n} p(x_i, y_i) \).
Jointly Typical Sequences: Picture

- Set of all jointly typical pairs of sequences: $2^{nH(X,Y)}$
- Set of all pairs of marginally typical sequences:
  - $2^{nH(X)}$
  - $2^{nH(Y)}$
  - $2^{nH(Y|X)}$
  - $2^{nH(X|Y)}$
Jointly Typical Sequences: Intuition

So intuitively,

\[
\frac{\text{num. jointly typical seqs.}}{\text{num ind. chosen typical seqs.}} = \frac{2^{nH(X,Y)}}{2^{nH(X)}2^{nH(Y)}} = 2^{n(H(X,Y)-H(X)-H(Y))} = 2^{-nI(X;Y)}
\]  

(14.14)  
(14.15)  
(14.16)
Jointly Typical Sequences: Intuition

- So intuitively,

\[
\frac{\text{num. jointly typical seqs.}}{\text{num ind. chosen typical seqs.}} = \frac{2^{nH(X,Y)}}{2^{nH(X)}2^{nH(Y)}} = 2^{n(H(X,Y) - H(X) - H(Y))} = 2^{-nI(X;Y)} \quad (14.14, 14.15, 14.16)
\]

- So if we independently at random choose two (singly) typical sequences for \( X \) and \( Y \), then the chance that it will be an \((X, Y)\) jointly typical sequence decreases exponentially with \( n \), as long as \( I(X;Y) > 0 \).
Jointly Typical Sequences: Intuition

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(14.14) (14.15) (14.16)

- So if we independently at random choose two (singly) typical sequences for \(X\) and \(Y\), then the chance that it will be an \((X, Y)\) jointly typical sequence decreases exponentially with \(n\), as long as \(I(X; Y) > 0\).

- to decrease this chance as much as possible, it can become \(2^{-nC}\).
Joint AEP

Theorem 14.5.1

Let \((X^n, Y^n) \sim p(x^n, y^n) = \prod_{i=1}^{n} p(x_i, y_i)\). Then
Theorem 14.5.1

Let \((X^n, Y^n) \sim p(x^n, y^n) = \prod_{i=1}^{n} p(x_i, y_i)\). Then

1. \(\Pr\left( (X^n, Y^n) \in A_c^{(n)} \right) \to 1 \text{ as } n \to \infty.\)
Joint AEP

**Theorem 14.5.1**

Let \((X^n, Y^n) \sim p(x^n, y^n) = \prod_{i=1}^n p(x_i, y_i)\). Then

1. \(\Pr\left((X^n, Y^n) \in A_{\epsilon}^{(n)}\right) \to 1\) as \(n \to \infty\).

2. \(|A_{\epsilon}^{(n)}| \leq 2^{n(H(X,Y)+\epsilon)}\) and \((1 - \epsilon)2^{n(H(X,Y)-\epsilon)} \leq |A_{\epsilon}^{(n)}|\).
Joint AEP

Theorem 14.5.1

Let \( (X^n, Y^n) \sim p(x^n, y^n) = \prod_{i=1}^{n} p(x_i, y_i) \). Then

1. \( \Pr \left( (X^n, Y^n) \in A_{\epsilon}^{(n)} \right) \to 1 \) as \( n \to \infty \).

2. \( |A_{\epsilon}^{(n)}| \leq 2^n(H(X,Y) + \epsilon) \) and \( (1 - \epsilon)2^n(H(X,Y) - \epsilon) \leq |A_{\epsilon}^{(n)}| \).

3. If \( (\tilde{X}^n, \tilde{Y}^n) \sim p(x^n)p(y^n) \) are drawn independently, then

\[
\Pr \left( (\tilde{X}^n, \tilde{Y}^n) \in A_{\epsilon}^{(n)} \right) \leq 2^{-n(I(X;Y) - 3\epsilon)} \tag{14.17}
\]

and for sufficiently large \( n \), we have

\[
\Pr \left( (\tilde{X}^n, \tilde{Y}^n) \in A_{\epsilon}^{(n)} \right) \geq (1 - \epsilon)2^{-n(I(X;Y) + 3\epsilon)} \tag{14.18}
\]
Joint AEP

**Theorem 14.5.1**

Let \((X^n, Y^n) \sim p(x^n, y^n) = \prod_{i=1}^{n} p(x_i, y_i)\). Then

1. \(\Pr \left( (X^n, Y^n) \in A_{\epsilon}^{(n)} \right) \to 1 \text{ as } n \to \infty.\)

2. \(|A_{\epsilon}^{(n)}| \leq 2^{n(H(X,Y)+\epsilon)} \text{ and } (1 - \epsilon)2^{n(H(X,Y)-\epsilon)} \leq |A_{\epsilon}^{(n)}|\).

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\]

and for sufficiently large \(n\), we have
\[
\Pr \left( (\tilde{X}^n, \tilde{Y}^n) \in A_{\epsilon}^{(n)} \right) \geq (1 - \epsilon)2^{-n(I(X;Y)+3\epsilon)} \tag{14.18}
\]

- Key property: we have bound on the probability of independently drawn sequences being jointly typical, falls off exponentially fast with \(n\), if \(I(X;Y) > 0\).
Joint AEP proof

Proof of \( \Pr \left( (X^n, Y^n) \in A^{(n)}_\epsilon \right) \to 1. \)
Proof of $\Pr \left( (X^n, Y^n) \in A^{(n)}_\epsilon \right) \to 1$. 

We have, by the w.l.l.n.s,

$$\frac{1}{n} \log \Pr(X^n) \to -E(\log p(X)) = H(X) \quad (14.19)$$
Proof of $\Pr \left( (X^n, Y^n) \in A^{(n)}_\epsilon \right) \to 1$.

- We have, by the w.l.l.n.s,

$$\frac{1}{n} \log \Pr(X^n) \to -E(\log p(X)) = H(X)$$  \hspace{1cm} (14.19)

so $\forall \epsilon > 0$, $\exists m_1$ such that for $n > m_1$

$$\Pr \left( \left| -\frac{1}{n} \log \Pr(X^n) - H(X) \right| > \epsilon \right) < \frac{\epsilon}{3}$$  \hspace{1cm} (14.20)

call this $S_1$
Joint AEP proof

Proof of \( \Pr \left( (X^n, Y^n) \in A^{(n)}_{\epsilon} \right) \to 1. \)

- We have, by the w.l.l.n.s,

\[
- \frac{1}{n} \log \Pr(X^n) \to -E(\log p(X)) = H(X)
\]

so \( \forall \epsilon > 0, \exists m_1 \) such that for \( n > m_1 \)

\[
\Pr \left( \left| - \frac{1}{n} \log \Pr(X^n) - H(X) \right| > \epsilon \right) < \frac{\epsilon}{3}
\]

(call this \( S_1 \))

- So, \( S_1 \) is a non-typical event.
Joint AEP proof

Proof of \( \Pr \left( (X^n, Y^n) \in A^n_e \right) \to 1. \)
Joint AEP proof

Proof of \( \Pr \left( (X^n, Y^n) \in A^{(n)}_\epsilon \right) \to 1 \).

Also, \( \exists m_2, m_3 \) such that \( \forall n > m_2 \), we have

\[
\Pr \left( \left| -\frac{1}{n} \log \Pr(Y^n) - H(Y) \right| > \epsilon \right) < \epsilon/3 \tag{14.21}
\]

This is called \( S_2 \). \( S_2 \) is a non-\( y \)-typical event.
Joint AEP proof

Proof of \( \Pr \left( (X^n, Y^n) \in A_\epsilon^{(n)} \right) \rightarrow 1. \)

- Also, \( \exists m_2, m_3 \) such that \( \forall n > m_2 \), we have

\[
\Pr \left( \left| -\frac{1}{n} \log \Pr(Y^n) - H(Y) \right| > \epsilon \right) < \epsilon/3 \tag{14.21}
\]

and call this \( S_2 \)

- and \( \forall n > m_3 \), we have

\[
\Pr \left( \left| -\frac{1}{n} \log \Pr(X^n, Y^n) - H(X, Y) \right| > \epsilon \right) < \epsilon/3 \tag{14.22}
\]

and call this \( S_3 \)

\[
\ldots
\]
Joint AEP proof

Proof of $\Pr \left( (X^n, Y^n) \in A^{(n)}_\epsilon \right) \to 1$.

- Also, $\exists m_2, m_3$ such that $\forall n > m_2$, we have

$$
\Pr \left( \left| -\frac{1}{n} \log \Pr(Y^n) - H(Y) \right| > \epsilon \right) < \epsilon/3 \quad (14.21)
$$

call this $S_2$

and $\forall n > m_3$, we have

$$
\Pr \left( \left| -\frac{1}{n} \log \Pr(X^n, Y^n) - H(X, Y) \right| > \epsilon \right) < \epsilon/3 \quad (14.22)
$$

call this $S_3$

- So all events $S_1$, $S_2$ and $S_3$ are non-typical events.

...
Joint AEP proof

Proof of $\Pr\left(\left(X^n, Y^n\right) \in A_{\epsilon}^{(n)}\right) \rightarrow 1$. 

...
Joint AEP proof

Proof of $\Pr\left((X^n, Y^n) \in A^{(n)}_\epsilon\right) \to 1$.

- For $n > \max(m_1, m_2, m_3)$, we have that $p(S_1 \cup S_2 \cup S_3) \leq \epsilon = 3\epsilon/3$ by the union bound (or Boole’s inequality).
Proof of $\Pr \left( (X^n, Y^n) \in A^{(n)}_{\epsilon} \right) \to 1$.

- For $n > \max(m_1, m_2, m_3)$, we have that $p(S_1 \cup S_2 \cup S_3) \leq \epsilon = 3\epsilon/3$ by the union bound (or Boole’s inequality).

- So, non-typicality has probability $< \epsilon$, meaning $\Pr(A^{(n)c}_{\epsilon}) \leq \epsilon$ giving $\Pr(A^{(n)}_{\epsilon}) \geq 1 - \epsilon$, as desired. $\square$ for 1.

...
Joint AEP proof

Proof of $|A^{(n)}_{\epsilon}| \leq 2^n (H(X,Y)+\epsilon)$.
Joint AEP proof

Proof of $|A_\epsilon^{(n)}| \leq 2^n(H(X,Y)+\epsilon)$.

We have

$$1 = \sum_{x^n,y^n} p(x^n, y^n) \geq \sum_{(x^n,y^n) \in A_\epsilon^{(n)}} p(x^n, y^n) \geq |A_\epsilon^{(n)}| 2^{-n(H(X,Y)+\epsilon)}$$

(14.23)

$$\Rightarrow |A_\epsilon^{(n)}| \leq 2^n(H(X,Y)+\epsilon)$$

(14.24)
Joint AEP proof

Proof of $|A_{\epsilon}^{(n)}| \leq 2^n(H(X,Y)+\epsilon)$.

- We have

\[
1 = \sum_{x^n,y^n} p(x^n, y^n) \geq \sum_{(x^n,y^n) \in A_{\epsilon}^{(n)}} p(x^n, y^n) \geq |A_{\epsilon}^{(n)}| 2^{-n(H(X,Y)+\epsilon)}
\]  

\[ (14.23) \]

\[
\Rightarrow \ |A_{\epsilon}^{(n)}| \leq 2^n(H(X,Y)+\epsilon)
\]

\[ (14.24) \]

- Also, from before, $\Pr(A_{\epsilon}^{(n)}) \geq 1 - \epsilon$ for big $n$, giving:

\[
1 - \epsilon \leq \sum_{(x^n,y^n) \in A_{\epsilon}^{(n)}} p(x^n, y^n) \leq |A_{\epsilon}^{(n)}| 2^{-n(H(X,Y)-\epsilon)}
\]

\[ (14.25) \]

\[
\Rightarrow \ |A_{\epsilon}^{(n)}| \geq (1 - \epsilon)2^n(H(X,Y)-\epsilon)
\]

\[ (14.26) \]

\[ \Box \] for 2.

...
Joint AEP proof

Proof of two indep. sequences are likely not jointly typical.
Joint AEP proof

Proof of two indep. sequences are likely not jointly typical.

- Let $\tilde{X}^n, \tilde{Y}^n$ be independent $\sim p(x^n)p(y^n)$, i.e. the two sequences are independent of each other.

\[\begin{align*}
\tilde{X}^n & \perp \tilde{Y}^n
\end{align*}\]
Joint AEP proof

Proof of two indep. sequences are likely not jointly typical.

- Let $\tilde{X}^n, \tilde{Y}^n$ be independent $\sim p(x^n)p(y^n)$, i.e. the two sequences are independent of each other.

- Then we have the following two derivations:

\[
\Pr\left( (\tilde{X}^n, \tilde{Y}^n) \in A_{\epsilon}^{(n)} \right) = \sum_{(x^n,y^n) \in A_{\epsilon}^{(n)}} p(x^n)p(y^n) \leq 2^n(H(X,Y)+\epsilon)2^{-n(H(X)-\epsilon)}2^{-n(H(Y)-\epsilon)} = 2^{-n(I(X;Y)-3\epsilon)}.
\]
Proof of two indep. sequences are likely not jointly typical.

- Let \( \tilde{X}^n, \tilde{Y}^n \) be independent \( \sim p(x^n)p(y^n) \), i.e. the two sequences are independent of each other.

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\Pr \left( (\tilde{X}^n, \tilde{Y}^n) \in A^{(n)}_\epsilon \right) = \sum_{(x^n,y^n) \in A^{(n)}_\epsilon} p(x^n)p(y^n) \leq 2^n(H(X,Y) + \epsilon) 2^{-n(H(X) - \epsilon)} 2^{-n(H(Y) - \epsilon)} \quad (14.27)
\]

\[
 = 2^{-n(I(X;Y) - 3\epsilon)}. \quad (14.28)
\]

And since \( |A^{(n)}_\epsilon| \geq (1 - \epsilon)2^n(H(X,Y) - \epsilon) \),

\[
\Pr \left( (\tilde{X}^n, \tilde{Y}^n) \in A^{(n)}_\epsilon \right) \geq (1 - \epsilon)2^n(H(X,Y) - \epsilon) 2^{-n(H(X) + \epsilon)} 2^{-n(H(Y) + \epsilon)}
\]

\[
= (1 - \epsilon)2^{-n(I(X;Y) + 3\epsilon)} \quad (14.29)
\]
Another Intuitive (and somewhat redundant) Reprieve

- There are $\approx 2^{nH(X)}$ typical $X$ sequences
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- The total number of independent typical pairs is $\approx 2^{nH(X)}2^{nH(Y)}$. 
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- The fraction of independent typical sequences that are jointly typical is:

\[
\frac{2^{nH(X,Y)}}{2^{nH(X)}2^{nH(Y)}} = 2^n(H(X,Y)-H(X)-H(Y)) = 2^{-nI(X,Y)} \quad (14.31)
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$$\frac{2^{nH(X,Y)}}{2^{nH(X)}2^{nH(Y)}} = 2^n(H(X,Y) - H(X) - H(Y)) = 2^{-nI(X,Y)} \quad (14.31)$$

and this is essentially the probability that a randomly chosen pair of (marginally) typical sequences is jointly typical.
Jointly Typical Sequences: Picture

- Set of all jointly typical pairs of sequences: $2^{nH(X,Y)}$
- Set of all pairs of marginally typical sequences
- $2^{nH(X|Y)}$
- $2^{nH(Y|X)}$
- $2^{nH(Y)}$
- $2^{nH(X)}$
More Intuition

- So if we use typicality to decode (which we will) then there are about $2^{nI(X;Y)}$ pairs of sequences available before we start needing to use pairs that would be jointly typical if chosen randomly.
So if we use typicality to decode (which we will) then there are about $2^{nI(X;Y)}$ pairs of sequences available before we start needing to use pairs that would be jointly typical if chosen randomly.

Ex: if $p(x) = 1/M$ then we can choose about $M$ samples before we see a given particular $x$, on average.
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Also, the probability that some other $\hat{x}^n$ is jointly typical with $y^n$ is about $2^{-nI(X;Y)}$, where $I(X;Y)$ is the mutual information between $X$ and $Y$. 
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Given a received codeword $y^n$, find an $x^n$ that is jointly typical with $y^n$.

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so if we use $< 2^{nI(X;Y)}$ codewords, then some other sequence being jointly typical will occur with vanishingly small probability for large $n$. 
Channel Coding Theorem (Shannon 1948): more formally

Theorem 14.6.1

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**Theorem 14.6.1**

All rates below $C \triangleq \max_p p(x) I(X; Y)$ are achievable. Specifically, $\forall R < C$, there exists a sequence of $(2^{nR}, n)$ codes with maximum probability of error $\lambda^{(n)} \to 0$ as $n \to \infty$. 
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- Implications: as long as we do not code above capacity we can, for all intents and purposes, code with zero error.
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- This is true for all noisy channels representable under this model.
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- Implications: as long as we do not code above capacity we can, for all intents and purposes, code with zero error.
- This is true for all noisy channels representable under this model.
- We’re talking about discrete channels now, but we generalize this to continuous channels in the coming weeks and/or IT-II.
Channel Theorem

- We could look at error for a particular code and bound its errors.
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- Recall: idea is, for a given channel \((\mathcal{X}, p(y|x), \mathcal{Y})\) come up with a \((2^{nR}, n)\) code of rate \(R\) which means we need:
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**Channel Theorem**

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Recall: idea is, for a given channel $(\mathcal{X}, p(y|x), \mathcal{Y})$ come up with a $(2^{nR}, n)$ code of rate $R$ which means we need:

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Two parts to prove: 1) all rates $R < C$ are achievable (exists a code with vanishing error). Conversely, 2) if the error goes to zero, then must have $R < C$. 

Prof. Jeff Bilmes
EE514a/Fall 2019/Info. Theory I – Lecture 14 - Nov 18th, 2019
L14 F43/51 (pg.146/202)
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All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

- Given $R < C$, assume use of $p(x)$ and generate $2^{nR}$ random codewords using $p(x^n) = \prod_{i=1}^{n} p(x_i)$. 
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- Set of random codewords (the codebook) can be seen as a matrix:

$$C = \begin{bmatrix} x_1(1) & x_2(1) & x_3(1) & \cdots & x_n(1) \\ x_1(2) & x_2(2) & x_3(2) & \cdots & x_n(2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1(2^{nR}) & x_2(2^{nR}) & x_3(2^{nR}) & \cdots & x_n(2^{nR}) \end{bmatrix} \quad (14.32)$$
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  \vdots & \vdots & \vdots & \ddots & \vdots \\
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- Since $C$ is random, $\Pr(C)$ is sensible.
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- Since $C$ is random, $\Pr(C)$ is sensible.
- So, there are $2^{nR}$ codes each of length $n$ generated via $p(x)$.
- To send any message $\omega \in \{1, 2, \ldots, M = 2^{nR}\}$, we send codeword $x_{1:n}(\omega) = (x_1(\omega), x_2(\omega), \ldots, x_n(\omega))$. 

...
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

- Can compute probabilities of a given codeword for $\omega \in \{1, 2, \ldots, M\}$

\[
\Pr(x^n(\omega)) = \prod_{i=1}^{n} \Pr(x_i(\omega)), \quad \omega \in \{1, 2, \ldots, M\}
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(14.33)
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p(C) = \sum_{\omega} p(x^n(\omega))
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p(C) = \prod_{\omega=1}^{2^n R} \prod_{i=1}^{n} p(x_i(\omega))
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Proof that all rates $R < C$ are achievable.

Consider the following encoding/decoding scheme:

1. Generate a random codebook as above according to $p(x)$.
2. Codebook known to both sender/receiver (who also knows $p(y|x)$).
3. Generate messages $W$ according to the uniform distribution (we'll see why shortly), $p(W = w) = 2^{-nR}$, for $w = 1, \ldots, 2^{nR}$.
4. Send $x_n(w)$ over the channel.
5. Receiver receives $Y_n$ according to distribution $Y_n \sim p(y_n|x_n(w)) = \sum_{i=1}^{n} p(y_i|x_i(w))$. (14.35)
6. The signal is decoded using typical set decoding (to be described).
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Y^n \sim p(y^n|x^n(\omega)) = \prod_{i=1}^{n} p(y_i|x_i(\omega)) \tag{14.35}
\]
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

Consider the following encoding/decoding scheme:

1. Generate a random codebook as above according to $p(x)$.
2. Codebook known to both sender/receiver (who also knows $p(y|x)$).
3. Generate messages $W$ according to the uniform distribution (we’ll see why shortly), $p(W = \omega) = 2^{-nR}$, for $\omega = 1, \ldots, 2^{nR}$.
4. Send $x^n(\omega)$ over the channel.
5. Receiver receives $Y^n$ according to distribution

$$Y^n \sim p(y^n|x^n(\omega)) = \prod_{i=1}^{n} p(y_i|x_i(\omega))$$  \hspace{1cm} (14.35)

6. The signal is decoded using typical set decoding (to be described).
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

Typical set decoding: Decode message as $\hat{\omega}$ if

...
Proof that all rates $R < C$ are achievable.

Typical set decoding: Decode message as $\hat{\omega}$ if

1. $(x^n(\hat{\omega}), y^n)$ is jointly typical, i.e., $(x^n(\hat{\omega}), y^n) \in A_\epsilon^{(n)}$
Proof that all rates $R < C$ are achievable.

Typical set decoding: Decode message as $\hat{\omega}$ if

1. $(x^n(\hat{\omega}), y^n)$ is jointly typical, i.e., $(x^n(\hat{\omega}), y^n) \in A^{(n)}_{\epsilon}$

2. $\exists$ no other $k \neq \hat{\omega}$ s.t. $(x^n(k), y^n) \in A^{(n)}_{\epsilon}$ (i.e., $\hat{\omega}$ is unique)
All rates $R < C$ are achievable.

**Proof that all rates $R < C$ are achievable.**

Typical set decoding: Decode message as $\hat{\omega}$ if

1. $(x^n(\hat{\omega}), y^n)$ is jointly typical, i.e., $(x^n(\hat{\omega}), y^n) \in A^{(n)}_\epsilon$
2. $\exists$ no other $k \neq \hat{\omega}$ s.t. $(x^n(k), y^n) \in A^{(n)}_\epsilon$ (i.e., $\hat{\omega}$ is unique)

Otherwise output special invalid integer “0” (error).
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

Typical set decoding: Decode message as $\hat{w}$ if

1. $(x^n(\hat{w}), y^n)$ is jointly typical, i.e., $(x^n(\hat{w}), y^n) \in A^{(n)}_{\epsilon}$
2. $\exists$ no other $k \neq \hat{w}$ s.t. $(x^n(k), y^n) \in A^{(n)}_{\epsilon}$ (i.e., $\hat{w}$ is unique)

Otherwise output special invalid integer “0” (error). Three types of errors might occur (type A, B, or C).
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

Typical set decoding: Decode message as $\hat{\omega}$ if

1. $(x^n(\hat{\omega}), y^n)$ is jointly typical, i.e., $(x^n(\hat{\omega}), y^n) \in A^{(n)}_\epsilon$

2. $\exists$ no other $k \neq \hat{\omega}$ s.t. $(x^n(k), y^n) \in A^{(n)}_\epsilon$ (i.e., $\hat{\omega}$ is unique)

Otherwise output special invalid integer “0” (error). Three types of errors might occur (type A, B, or C).

A: $\exists k \neq \hat{\omega}$ s.t. $(x^n(k), y^n) \in A^{(n)}_\epsilon$ (i.e., $> 1$ possible typical message).
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

Typical set decoding: Decode message as $\hat{\omega}$ if

1. $(x^n(\hat{\omega}), y^n)$ is jointly typical, i.e., $(x^n(\hat{\omega}), y^n) \in A^{(n)}_{\epsilon}$
2. $\exists$ no other $k \neq \hat{\omega}$ s.t. $(x^n(k), y^n) \in A^{(n)}_{\epsilon}$ (i.e., $\hat{\omega}$ is unique)

Otherwise output special invalid integer “0” (error). Three types of errors might occur (type A, B, or C).

A: $\exists k \neq \hat{\omega}$ s.t. $(x^n(k), y^n) \in A^{(n)}_{\epsilon}$ (i.e., $> 1$ possible typical message).

B: no $\hat{\omega}$ s.t. $(x^n(\hat{\omega}), y^n)$ is jointly typical.
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

Typical set decoding: Decode message as $\hat{\omega}$ if

1. $(x^n(\hat{\omega}), y^n)$ is jointly typical, i.e., $(x^n(\hat{\omega}), y^n) \in A_{\epsilon}^{(n)}$

2. $\exists$ no other $k \neq \hat{\omega}$ s.t. $(x^n(k), y^n) \in A_{\epsilon}^{(n)}$ (i.e., $\hat{\omega}$ is unique)

Otherwise output special invalid integer “0” (error). Three types of errors might occur (type A, B, or C).

A: $\exists k \neq \hat{\omega}$ s.t. $(x^n(k), y^n) \in A_{\epsilon}^{(n)}$ (i.e., $> 1$ possible typical message).

B: no $\hat{\omega}$ s.t. $(x^n(\hat{\omega}), y^n)$ is jointly typical.

C: if $\hat{\omega} \neq \omega$, i.e., wrong codeword is jointly typical.
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

Typical set decoding: Decode message as $\hat{\omega}$ if

1. $(x^n(\hat{\omega}), y^n)$ is jointly typical, i.e., $(x^n(\hat{\omega}), y^n) \in A^{(n)}_\epsilon$

2. $\exists$ no other $k \neq \hat{\omega}$ s.t. $(x^n(k), y^n) \in A^{(n)}_\epsilon$ (i.e., $\hat{\omega}$ is unique)

Otherwise output special invalid integer “0” (error). Three types of errors might occur (type A, B, or C).

A: $\exists k \neq \hat{\omega}$ s.t. $(x^n(k), y^n) \in A^{(n)}_\epsilon$ (i.e., $> 1$ possible typical message).

B: no $\hat{\omega}$ s.t. $(x^n(\hat{\omega}), y^n)$ is jointly typical.

C: if $\hat{\omega} \neq \omega$, i.e., wrong codeword is jointly typical.

Note: maximum likelihood decoding is optimal, but typical set decoding is not, but this will be good enough to show the result.
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

(also) three types of quality measures we might be interested in.
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

(also) three types of quality measures we might be interested in.

1. Code specific error

\[ P_e^{(n)}(C) = \Pr(\hat{\omega} \neq \omega|C) = \frac{1}{2^n R} \sum_{i=1}^{2^n R} \lambda_i \quad (14.36) \]

where (as a reminder)

\[ \lambda_i = \Pr(g(y^n) \neq i|X^n = x^n(i)) = \sum_{y^n} p(y^n|x^n(i)) \mathbf{1}\{g(y^n) \neq i\} \]
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

(Also) three types of quality measures we might be interested in.

1. Code specific error

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but we would like something easier to analyze.
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

(also) three types of quality measures we might be interested in.

1. Code specific error

$$P_e^{(n)}(C) = \Pr(\hat{\omega} \neq \omega | C) = \frac{1}{2nR} \sum_{i=1}^{2nR} \lambda_i$$  \hspace{1cm} (14.36)

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but we would like something easier to analyze.

2. Average error over all randomly generated codes (avg. of avg.)

$$\Pr(\mathcal{E}) = \sum_C \Pr(C) \Pr(\hat{W} \neq W | C) = \sum_C \Pr(C) P_e(C)$$  \hspace{1cm} (14.37)

...
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

(and also) three types of quality measures we might be interested in.

1. Code specific error

$$P_e^{(n)}(C) = \Pr(\hat{\omega} \neq \omega | C) = \frac{1}{2^{nR}} \sum_{i=1}^{2^{nR}} \lambda_i$$  (14.36)

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$$\Pr(\mathcal{E}) = \sum_C \Pr(C) \Pr(\hat{W} \neq W | C) = \sum_C \Pr(C) P_e(C)$$  (14.37)

Surprisingly, this is much easier to analyze than $P_e$...
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

(Also) three types of quality measures we might be interested in.
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

(also) three types of quality measures we might be interested in.

- Max error of the code, ultimately what we want to use

$$P_{C, \max}(C) = \max_{i \in \{1, 2, ..., M\}} \lambda_i \quad (14.38)$$

We want to show that if $R < C$, then exists a codebook $C$ s.t. this error $\to 0$ (and that if $R > C$ error must $\to 1$).
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

(also) three types of quality measures we might be interested in.

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All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

(Also) three types of quality measures we might be interested in.

3. Max error of the code, ultimately what we want to use

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We want to show that if $R < C$, then exists a codebook $C$ s.t. this error $\to 0$ (and that if $R > C$ error must $\to 1$).

Our method is to:

1. Expand average error (bullet 2 above) and show that it is small.
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

(also) three types of quality measures we might be interested in.

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Our method is to:

1. Expand average error (bullet 2 above) and show that it is small.
2. deduce that $\exists$ at least 1 code with small error
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

(Also) three types of quality measures we might be interested in.

3. Max error of the code, ultimately what we want to use

\[ P_{C,\text{max}}(C) = \max_{i \in \{1,2,\ldots,M\}} \lambda_i \quad (14.38) \]

We want to show that if $R < C$, then exists a codebook $C$ s.t. this error $\to 0$ (and that if $R > C$ error must $\to 1$).

Our method is to:

1. Expand average error (bullet 2 above) and show that it is small.
2. Deduce that $\exists$ at least 1 code with small error
3. Show that this can be modified to have small maximum probability of error.
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

$$\Pr(\mathcal{E}) = \sum_{C} \Pr(C) P_{e}^{(n)}(C)$$

(14.40)

$$\Pr(\mathcal{E}) = \sum_{C} \Pr(C) P_{e}^{(n)}(C)$$

(14.41)
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

\[
\Pr(\mathcal{E}) = \sum_C \Pr(C) P_e^{(n)}(C) = \sum_C \Pr(C) \frac{1}{2^{nR}} \sum_{\omega=1}^{2^{nR}} \lambda_\omega(C)
\]

(14.39)

\[
\Pr(\mathcal{E}) \leq \frac{1}{2^{nR}} \sum_{\omega=1}^{2^{nR}} \lambda_\omega(C)
\]

(14.40)

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Proof that all rates $R < C$ are achievable.

$$\Pr(\mathcal{E}) = \sum_C \Pr(C) P^{(n)}_e(C) = \sum_C \Pr(C) \frac{1}{2^{nR}} \sum_{\omega=1}^{2^{nR}} \lambda_\omega(C)$$  \hspace{1cm} (14.39)

$$= \frac{1}{2^{nR}} \sum_{\omega=1}^{2^{nR}} \sum_C \Pr(C) \lambda_\omega(C)$$  \hspace{1cm} (14.40)

$$\Pr(\mathcal{E}) = \sum_C \Pr(C) P^{(n)}_e(C) = \sum_C \Pr(C) \frac{1}{2^{nR}} \sum_{\omega=1}^{2^{nR}} \lambda_\omega(C)$$  \hspace{1cm} (14.39)

$$= \frac{1}{2^{nR}} \sum_{\omega=1}^{2^{nR}} \sum_C \Pr(C) \lambda_\omega(C)$$  \hspace{1cm} (14.40)

\[
\text{(14.41)}
\]
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

\[
\Pr(\mathcal{E}) = \sum_C \Pr(C) P_{e}(n)(C) = \sum_C \Pr(C) \frac{1}{2^{nR}} \sum_{\omega=1}^{2^{nR}} \lambda_\omega(C)
\]  

(14.39)

\[
= \frac{1}{2^{nR}} \sum_{\omega=1}^{2^{nR}} \sum_C \Pr(C) \lambda_\omega(C)
\]  

(14.40)

but

\[
(14.41)
\]
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

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\Pr(\mathcal{E}) = \sum_{C} \Pr(C) P_{e}^{(n)}(C) = \sum_{C} \Pr(C) \frac{1}{2^{nR}} \sum_{\omega=1}^{2^{nR}} \lambda_{\omega}(C)
\]

\[
= \frac{1}{2^{nR}} \sum_{\omega=1}^{2^{nR}} \sum_{C} \Pr(C) \lambda_{\omega}(C)
\]

but

\[
\sum_{C} \Pr(C) \lambda_{\omega}(C)
\]

(14.41)
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

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\Pr(\mathcal{E}) = \sum_{C} \Pr(C) P_{e}^{(n)}(C) = \sum_{C} \Pr(C) \frac{1}{2^{nR}} \sum_{\omega=1}^{2^{nR}} \lambda_\omega(C) \tag{14.39}
\]

\[
= \frac{1}{2^{nR}} \sum_{\omega=1}^{2^{nR}} \sum_{C} \Pr(C) \lambda_\omega(C) \tag{14.40}
\]

but

\[
\sum_{C} \Pr(C) \lambda_\omega(C) = \sum_{C} \Pr(g(Y^n) \neq \omega | X^n = x^n(\omega)) \Pr(x^n(1), \ldots, x^n(2^{nR})) \tag{14.41}
\]
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

\[
\Pr(\mathcal{E}) = \sum_C \Pr(C) P_e^{(n)}(C) = \sum_C \Pr(C) \frac{1}{2^{nR}} \sum_{\omega=1}^{2^nR} \lambda_\omega(C) \\
= \frac{1}{2^{nR}} \sum_{\omega=1}^{2^nR} \sum_C \Pr(C) \lambda_\omega(C) \\
\] (14.39)

but

\[
\sum_C \Pr(C) \lambda_\omega(C) = \sum_C \Pr(g(Y^n) \neq \omega | X^n = x^n(\omega)) \Pr(x^n(1), \ldots, x^n(2^{nR})) \\
\] (14.41)
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

$$\Pr(\mathcal{E}) = \sum_{C} \Pr(C) P_e^{(n)}(C) = \sum_{C} \Pr(C) \frac{1}{2^{nR}} \sum_{\omega=1}^{2^nR} \lambda_{\omega}(C)$$ (14.39)

$$= \frac{1}{2^{nR}} \sum_{\omega=1}^{2^nR} \sum_{C} \Pr(C) \lambda_{\omega}(C)$$ (14.40)

but

$$\sum_{C} \Pr(C) \lambda_{\omega}(C) = \sum_{C} \Pr(g(Y^n) \neq \omega | X^n = x^n(\omega)) \Pr(x^n(1), \ldots, x^n(2^{nR}))$$ (14.41)

...
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

\[
\Pr(E) = \sum_C \Pr(C) P_e^{(n)}(C) = \sum_C \Pr(C) \frac{1}{2^{nR}} \sum_{\omega=1}^{2^{nR}} \lambda_\omega(C) \\
= \frac{1}{2^{nR}} \sum_{\omega=1}^{2^{nR}} \sum_C \Pr(C) \lambda_\omega(C) 
\]

but

\[
\sum_C \Pr(C) \lambda_\omega(C) = \sum_C \Pr(g(Y^n) \neq \omega | X^n = x^n(\omega)) \Pr(x^n(1), \ldots, x^n(2^{nR})) \\
= \sum_{x^n(1), x^n(2), \ldots, x^n(2^{nR})} \text{stuff} 
\]
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

$$\sum_C \Pr(C) \lambda_\omega(C)$$

(14.42)

Last sum is same regardless of $!c a l l i t$. Thus, we can arbitrarily assume that $!=1$.

...
All rates \( R < C \) are achievable.

Proof that all rates \( R < C \) are achievable.

\[
\sum_C \Pr(C) \lambda_\omega(C) = \sum_{x^n(1), \ldots, x^n(\omega-1), \ldots, x^n(\omega+1), \ldots, x^n(2^nR)} p \left( \frac{x^n(1), \ldots, x^n(\omega-1), \ldots, x^n(\omega+1), \ldots, x^n(2^nR)}{x^n(\omega)} \right) \sum_{x^n(\omega)} \Pr(g(Y^n) \neq \omega | X^n = x^n(\omega)) \Pr(x^n(\omega))
\]

(14.42)

\[
\sum_{x^n(\omega)} \Pr(g(Y^n) \neq \omega | X^n = x^n(\omega)) \Pr(x^n(\omega)) = \sum_{C} \Pr(C) \lambda_\omega(C)
\]

(14.44)
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

$$\sum_{C} \Pr(C) \lambda_{\omega}(C)$$

$$= \sum_{x^n(1), \ldots, x^n(\omega-1), x^n(\omega+1), \ldots, x^n(2nR)} p\left(\frac{x^n(1), \ldots, x^n(\omega-1), x^n(\omega+1), \ldots, x^n(2nR)}{x^n(\omega)}\right) \sum_{x^n(\omega)} \Pr(g(Y^n) \neq \omega | X^n = x^n(\omega)) \Pr(x^n(\omega))$$

$$\prod_{i \neq \omega} \Pr(x^n(i))$$

(14.42)
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

\[
\sum_C \Pr(C) \lambda_\omega(C) \\
= \sum_{\omega} \left\{ \prod_{i \neq \omega} \Pr(x^n(i)) \right\} \sum_{x^n(\omega)} \Pr(g(Y^n) \neq \omega | X^n = x^n(\omega)) \Pr(x^n(\omega))
\]

\[
= 1
\]

(14.42)

(14.44)
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

\[
\sum_C \Pr(C) \lambda_\omega(C) \tag{14.42}
\]

\[
= \sum_{x^n(1),...,x^n(\omega-1), \atop x^n(\omega+1),...,x^n(2nR)} p\left(\frac{x^n(1),...,x^n(\omega-1), \atop x^n(\omega+1),...,x^n(2nR)}{x^n(1),...,x^n(\omega-1), \atop x^n(\omega+1),...,x^n(2nR)}\right) \sum_{x^n(\omega)} \Pr(g(Y^n) \neq \omega | X^n = x^n(\omega)) \Pr(x^n(\omega))
\]

\[
= 1 \tag{14.43}
\]

\[
= \sum_{x^n(\omega)} \Pr(g(Y^n) \neq \omega | X^n = x^n(\omega)) \Pr(x^n(\omega)) \tag{14.44}
\]

...
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

\[
\sum_{C} \Pr(C) \lambda_{\omega}(C)
\]

\[
= \sum_{C} \left\{ \prod_{i \neq \omega} \Pr(x^n(i)) \right\} \sum_{x^n(\omega)} \Pr(g(Y^n) \neq \omega|X^n = x^n(\omega)) \Pr(x^n(\omega))
\]

\[
= \sum_{x^n(\omega)} \Pr(g(Y^n) \neq \omega|X^n = x^n(\omega)) \Pr(x^n(\omega))
\]

\[
= \sum_{x^n \in \mathcal{X}^n} \Pr(g(Y^n) \neq 1|X^n = x^n(1)) \Pr(x^n(1))
\]
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

\[
\sum_C \Pr(C) \lambda_\omega(C) = \sum_{x^n(\omega)} \Pr(g(Y^n) \neq \omega \mid X^n = x^n(\omega)) \Pr(x^n(\omega)) \]

(14.43)

\[
= \sum_{x^n(\omega) \in \mathcal{X}^n} \Pr(g(Y^n) \neq 1 \mid X^n = x^n(1)) \Pr(x^n(1)) = \sum_C \Pr(C) \lambda_1(C)
\]

(14.44)
All rates $R < C$ are achievable.

Proof that all rates $R < C$ are achievable.

$$\sum_C \Pr(C) \lambda_\omega(C)$$  \hspace{1cm} (14.42)

\[
= \sum_{x^n(1), \ldots, x^n(\omega-1), \ldots, x^n(2nR)} \Pr(x^n(\omega)) = 1
\]

\[
= \sum_{x^n(\omega)} \Pr(g(Y^n) \neq \omega | X^n = x^n(\omega)) \Pr(x^n(\omega))
\]  \hspace{1cm} (14.43)

\[
= \sum_{x^n \in \mathcal{X}^n} \Pr(g(Y^n) \neq 1 | X^n = x^n(1)) \Pr(x^n(1)) = \sum_C \Pr(C) \lambda_1(C) = \beta
\]  \hspace{1cm} (14.44)
**All rates** \( R < C \) **are achievable.**

**Proof that all rates** \( R < C \) **are achievable.**

\[
\sum_C \Pr(C) \lambda_\omega(C) = \sum \left\{ \prod_{i \neq \omega} \Pr(x^n(i)) \right\} \sum \Pr(g(Y^n) \neq \omega \mid X^n = x^n(\omega)) \Pr(x^n(\omega))
\]

\[= \sum \Pr(g(Y^n) \neq \omega \mid X^n = x^n(\omega)) \Pr(x^n(\omega))
\]

\[= \sum_{x^n(\omega)} \sum_{x^n(x^n(\omega))} \Pr(g(Y^n) \neq 1 \mid X^n = x^n(1)) \Pr(x^n(1)) = \sum_C \Pr(C) \lambda_1(C) = \beta
\]

Last sum is same regardless of \( \omega \), call it \( \beta \). Thus, we can can arbitrarily assume that \( \omega = 1 \).