Class Road Map - IT-I

L1 (9/25): Overview, Communications, Information, Entropy
L2 (9/30): Entropy, Mutual Information, KL-Divergence
L3 (10/2): More KL, Jensen, more Venn, Log Sum, Data Proc. Inequality
L4 (10/7): Data Proc. Ineq., thermodynamics, Stats, Fano,
L5 (10/9): M. of Conv, AEP,
L6 (10/14): AEP, Source Coding, Types
L7 (10/16): Types, Univ. Src Coding, Stoc. Procs, Entropy Rates
L8 (10/21): Entropy rates, HMMs, Coding
L9 (10/23): Kraft ineq., Shannon Codes, Kraft ineq. II, Huffman

L10 (10/28):
L11 (10/30):
L12 (11/4):
LXX (11/6): In class midterm exam
LXX (11/11): Veterans Day holiday
L13 (11/13):
L14 (11/18):
L15 (11/20):
L16 (11/25):
L17 (11/27):
L18 (12/2):
L19 (12/4):
LXX (12/10): Final exam

Finals Week: December 9th–13th.
Read chapters 1 and 2 in our book (Cover & Thomas, “Information Theory”) (including Fano’s inequality).

Read chapters 3 and 4 in our book (Cover & Thomas, “Information Theory”).

Read sections 11.1 through 11.3 in our book (Cover & Thomas, “Information Theory”).

Read chapter 4 in our book (Cover & Thomas, “Information Theory”).
Homework

- Homework 1 on our assignment dropbox (https://canvas.uw.edu/courses/1319497/assignments), was due Tuesday, Oct 8th, 11:55pm.
- Homework 2 on our assignment dropbox (https://canvas.uw.edu/courses/1319497/assignments), due Friday 10/18/2019, 11:45pm.
- Homework 3 on our assignment dropbox (https://canvas.uw.edu/courses/1319497/assignments), due Tuesday 10/29/2019, 11:45pm.
What is entropy of this random walk

- So, the entropy of the random walk is

\[ H(\mathcal{X}) = (\text{overall edge uncertainty}) \]
\[ - (\text{overall node uncertainty in stationary condition}) \]

- Intuition: As node entropy decreases while keeping edge uncertainty constant, fewer nodes are hubs. Hubs that remain are widely connected (since edge entropy is fixed).

- In such case (few well connected hubs), it is likely one will land on such a hub (in a random walk) and then will be faced with a wide variety of choice as to where to go next \(\Rightarrow\) increase in overall uncertainty \(H(\mathcal{X}) = H(X_t|X_{t-1})\) of the walk.

- If node entropy goes up with edge entropy fixed, many nodes are hubs all with relatively low connectivity, hitting them doesn’t provide much choice \(\Rightarrow\) random walk entropy \(H(\mathcal{X}) = H(X_t|X_{t-1})\) goes down.
Practical Coding

- We want to develop practical coding algorithms that still approach, or achieve, the entropy limit.
- They might use the distribution $p(x)$ which is either given or is estimated in some way.
- We won’t get into any details on how to estimate $p(x)$ (that is a density estimation problem) but we assume we either have it or some approximation.
- We will ultimately look, however, at what happens if the true distribution is $p(x)$ and we use $q(x)$ instead.
Source Code

Definition 9.2.2 (source code)

A source code $C$ for r.v. $X$ is a mapping

$$C : \mathcal{X} \rightarrow \mathcal{D}^* = \{\mathcal{D} \cup (\mathcal{D} \times \mathcal{D}) \cup (\mathcal{D} \times \mathcal{D} \times \mathcal{D}) \cup \ldots\}$$

(9.24)

from $\mathcal{X}$ to $\mathcal{D}^*$, the set of finite strings from a $D$-ary alphabet. $C(x)$ is the codeword corresponding to $x$, and $\ell(x)$ is the length of the codeword.

Example 9.2.3

Let $\mathcal{X} = \{\text{red}, \text{blue}\}$. Then a code might be $C(\text{red}) = 00$ and $C(\text{blue}) = 11$, which would be a binary code for $\mathcal{D} = \{0, 1\}$.

Definition 9.2.4 (expected length)

The expected length $L(C)$ of code $C$ for r.v. $X$ with distribution $p(x)$ is

$$L(C) = \sum_x p(x) \ell(x)$$

(9.25)
Definition 9.2.3 (non-singular)

A code is said to be non-singular if every element of the range of $X$ (i.e., all elements of $\mathcal{X}$) maps to a different string in $\mathcal{D}^*$. I.e.,

$$x_i \neq x_j \Rightarrow C(x_i) \neq C(x_j)$$ (9.24)

- We can view this as a mapping. It is less strict than onto but sufficient for being able to decode individual symbols.

- Note that $C_I$ above is singular.
Our goal, and definition of Code Extension

- Before going further, note: our goal is to send or store a sequence of code words for a sequence of symbols.
- A non-singular code could be unique if $\exists$ a comma between code words (e.g., Morse code is such that there is a space).
- In general, however, it is better to have a self punctuating or instantaneous code.

**Definition 9.2.3 (code extension)**

A code extension $C^*$ of $C$ is a mapping from finite length strings of $D$, defined as:

$$C^*(x_1, x_2, \ldots, x_n) = C(x_1)C(x_2)\ldots C(x_n) \quad (9.24)$$

- Note that there are no commas in the extension, rather concatenation.
- Ex: If $C(x_1) = 0$ and $C(x_2) = 1$ then $C(x_1, x_2) = 01$. 
Definition 9.2.3 (uniquely decodable)

A code $C$ with extension $C^*$ is uniquely decodable if the extension $C^*$ is non-singular.

- $C_1$ singular. Extension of $C_1$ singular so $C_1$ not uniquely decodable.
- But how long must we wait until we know the source? In some even uniquely decodable cases, we might need to wait until the end.
- Ex: consider the code

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C(x)$</td>
<td>10</td>
<td>00</td>
<td>11</td>
<td>110</td>
</tr>
</tbody>
</table>

Is this code uniquely decodable? Yes.
Prefix codes

Definition 9.2.3 (prefix code)
A code is called a **prefix code** or an **instantaneous code** if no codeword is a prefix of any other codeword.

- We know the end of a codeword because it can’t be a prefix of any other codeword.
- Code in previous page is not prefix free, 11 was a prefix of 110 so we couldn’t decide between 11 or 110 until we could count the number of zeros.
- A prefix code is self-punctuating (since there are implicit punctuation marks between codewords).
- Prefix code $\Rightarrow$ uniquely decodable. But (as we saw) uniquely decodable $\not\Rightarrow$ prefix code.
Goal is to find a code with the shortest possible expected length.

From the above code class, we might think that we want to use codes from the largest class possible (since we might think we’re more likely to get shorter codes).

We can do better than entropy with non-singular codes, but we want lossless encoding $x = \text{ungzip}(\text{gzip}(x))$. 
Theorem 9.3.1 (Kraft inequality)

*For any instantaneous code (prefix code) over alphabet of size $D$, the codeword lengths $\ell_1, \ell_2, \ldots, \ell_m$ must satisfy*

$$\sum_{i} D^{-\ell_i} \leq 1 \quad (9.1)$$
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Conversely, given a set of codeword lengths satisfying the above inequality, $\exists$ an instantaneous code with these word lengths.
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- Note: converse says there exists a code with these lengths, not that all codes with these lengths will satisfy the inequality.
Kraft inequality

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- Key point: for $\ell_i$ satisfying Kraft, no further restriction imposed by also wanting a prefix code, so we might as well use a prefix code (assuming it is easy to find given the lengths)
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- Key point: for $\ell_i$ satisfying Kraft, no further restriction imposed by also wanting a prefix code, so we might as well use a prefix code (assuming it is easy to find given the lengths)
- Connects code existence to mathematical property on lengths!

Given Kraft lengths, can construct an instantaneous code (as we will see). Given lengths, can compute $E[\ell]$ and compare with $H$. 
proof of Kraft inequality.

- Represent set of codes on a $D$-ary (not necessarily balanced) tree:

...
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  ![Diagram of a D-ary tree representing the set of codes with codewords corresponding to leaves and the prefix condition ensuring no descendant of a codeword is a codeword.]
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- Represent set of codes on a $D$-ary (not necessarily balanced) tree:

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![D-ary tree diagram]

- Codewords correspond to leaves
- Path from root to leaf determines a codeword
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- Represent set of codes on a $D$-ary (not necessarily balanced) tree:

- Codewords correspond to leaves
- Path from root to leaf determines a codeword
- Prefix condition: won’t get to a codeword until we get to a leaf (no descendants of codewords are codewords)
Kraft inequality

... proof of Kraft inequality cont.

- $\ell_{\text{max}} = \max_i (\ell_i)$ is the length of the longest codeword.
Kraft inequality

... proof of Kraft inequality cont.

- $\ell_{\text{max}} = \max_i(\ell_i)$ is the length of the longest codeword.
- We can expand the full-tree down to depth $\ell_{\text{max}}$. 

Some nodes at that level $\ell_{\text{max}}$ are either:
- 1 codeword,
- 2 descendants of codewords, or
- 3 neither

Consider a codeword $i$ at level $\ell_i$ in tree (so it has length $\ell_i$). Then, there are $D_{\ell_{\text{max}} - \ell_i}$ descendants in the tree at level $\ell_{\text{max}}$. Because of prefix condition, descendants of code $i$ at level $\ell_i$ are disjoint from descendants of code $j$ at level $\ell_j$ when $i \neq j$ (i.e., descendant sets for different codewords are disjoint). Also, total number of nodes in set of all descendants is $\leq D_{\ell_{\text{max}}}$.
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- Also, total number of nodes in set of all descendants is $\leq D^{\ell_{\text{max}}}$.
Kraft inequality

... proof of Kraft inequality cont.

□ All of the above implies:

\[
\sum_{i} D^{\ell_{\text{max}} - \ell_i} \leq D^{\ell_{\text{max}}} \quad \Rightarrow \quad \sum_{i} D^{-\ell_i} \leq 1 \quad (9.2)
\]

Conversely: given codeword lengths \(\ell_1, \ell_2, \ldots, \ell_m\) satisfying Kraft inequality (we must construct a prefix code with these lengths).

Consider a full \(D\)-ary tree of depth \(\ell_{\text{max}}\) with \(D^{\ell_{\text{max}}}\) terminal nodes.

@ level 0, \(\exists\) fraction 1 of the descendants at each node at that level;
@ level 1, \(\exists\) fraction \(\frac{1}{D}\) descendants at each node at that level;
@ level 2, \(\exists\) fraction \(\frac{1}{D^2}\) . . .

In general, at each level \(i \in [0, \ell_{\text{max}}]\) in tree, there is a fraction \(D^{-\ell_i}\) terminal nodes that are descendants that stem from each of the \(D^{\ell_i}\) nodes at level \(i\). . .
Kraft inequality

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  - @ level 2, \( \exists \) fraction \( 1/D^2 \) ... 

- In general, at each level \( i \in [0, \ell_{\text{max}}] \) in tree, there is a fraction \( D^{-i} \) terminal nodes that are descendants that stem from each of the \( D^i \) nodes at level \( i \). 

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Sort the lengths \((\ell_1, \ell_2, \ldots, \ell_m)\) ascending to \((s_1, s_2, \ldots, s_m)\) with \(s_1 \leq s_2 \leq \cdots \leq s_m\). Note there are as many lengths as there are codewords.
Kraft inequality

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- For length \(s_1\) chose any node at level \(s_1\) to indicate the codeword.
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- For length \(s_1\) chose any node at level \(s_1\) to indicate the codeword.
- To ensure prefix free property, the node becomes a terminal node, thus eliminating a fraction \(D^{-s_1}\) of the terminal nodes at depth \(\ell_{\text{max}}\) (which would have been potential code words of longer length, but now they are out of the running).
Kraft inequality

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- Next: chose any remaining node at level \(s_2\) (we have \((D^{s_1} - 1)D^{s_2-s_1} > 0\) choices at this point) for next codeword, thus eliminating a fraction \(D^{-s_2}\) of the nodes

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proof of Kraft inequality cont.

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For length $s_1$ choose any node at level $s_1$ to indicate the codeword.

To ensure prefix free property, the node becomes a terminal node, thus eliminating a fraction $D^{-s_1}$ of the terminal nodes at depth $\ell_{\text{max}}$ (which would have been potential code words of longer length, but now they are out of the running).

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Total fraction eliminated is $D^{-s_1} + D^{-s_2}$.
Continuing this process, we eliminate a fraction $\sum_{i=1}^{m} D^{-s_i}$ of the nodes, while retaining that the code is instantaneous (a codeword can’t be a prefix of another).
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But since by assumption $\sum_{i=1}^{m} D^{-s_i} \leq 1$ we never eliminate more than all of the codewords, so this process won’t run out of codewords.

Thus, we have created a prefix-free code with the desired lengths.
Theorem 9.3.2 (countably infinite Kraft)

For any countably infinite set of codewords that form a prefix set, this satisfies the extended Kraft inequality, i.e.

\[
\sum_{i=1}^{\infty} D^{-\ell_i} \leq 1 \tag{9.3}
\]

Conversely, given \( \ell_i \) satisfying the above, \( \exists \) a prefix code with these lengths.

proof of countably infinite Kraft.

- Assume we have such a prefix code, and let the \( D \)-ary alphabet be \( \{0, 1, \ldots, D - 1\} \).
Infinite Kraft

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Conversely, given \( \ell_i \) satisfying the above, \( \exists \) a prefix code with these lengths.

**proof of countably infinite Kraft.**

- Assume we have such a prefix code, and let the \( D \)-ary alphabet be \( \{0, 1, \ldots, D-1\} \).
- Consider the \( i^{th} \) codeword \( y_1, y_2, \ldots, y_{\ell_i} \).
Kraft inequality

... proof of infinite Kraft.

- Consider expansion of codeword using fractional digits:

\[ 0.y_1y_2y_3 \ldots y_{\ell_i} = \sum_{j=1}^{\ell_i} y_j D^{-j} \]  

(9.4)
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Examples: When \(D = \{0, 1\}\) then 0.1 = 1/2, 0.01 = 1/4, 0.11 = 3/4, and 0.001 = 1/8 (so bits are after the binary point).
Kraft inequality

. . . proof of infinite Kraft.

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- Associate each codeword \( y_1:y_{\ell_i} \) with the half-open interval on the real line \([0.y_1y_2 \ldots y_{\ell_i}, 0.y_1y_2 \ldots y_{\ell_i} + 1/D^{\ell_i})\)
Kraft inequality

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Consider expansion of codeword using fractional digits:

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Examples: When \( D = \{0, 1\} \) then \( 0.1 = 1/2 \), \( 0.01 = 1/4 \), \( 0.11 = 3/4 \), and \( 0.001 = 1/8 \) (so bits are after the binary point).

Associate each codeword \( y_1:\ell_i \) with the half-open interval on the real line \( [0.y_1y_2 \ldots y_{\ell_i} , 0.y_1y_2 \ldots y_{\ell_i} + 1/D^{\ell_i}) \)

Example: With \( D = 10 \), then if \( 0.y_1y_2y_3 = 0.157 \), the associated half-open interval is \( [0.157, 0.158) \), and if \( 0.y_1y_2y_3 = 0.159 \), the associated half-open interval is \( [0.159, 0.160) \)
Consider expansion of codeword using fractional digits:

\[ 0.y_1y_2y_3 \ldots y_{\ell_i} = \sum_{j=1}^{\ell_i} y_j D^{-j} \] (9.4)

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\[ 0.y_1y_2y_3 \ldots y_{\ell_i} \]

\[ \overbrace{\ldots \bullet \circ \bullet \circ \bullet \circ \circ \circ \ldots}^\text{...} \]
So the interval for codeword $y_1y_2y_3\ldots y_\ell_i$ corresponds to the set of all real numbers that begins with $0.y_1y_2y_3\ldots y_\ell_i$. 

Length of interval for codeword $y_1y_2y_3\ldots y_\ell_i$ is $D - \ell_i$. And since all intervals live in $[0,1)$ we must have 

$$\sum i (D - \ell_i) \leq 1 \quad (9.5)$$

Proof of converse is similar to finite case and also to arithmetic coding that we’ll soon see, so we skip the proof here.
Kraft inequality

...proof of infinite Kraft.

- So the interval for codeword $y_1 y_2 y_3 \ldots y_{\ell_i}$ corresponds to the set of all real numbers that begins with $0.y_1 y_2 y_3 \ldots y_{\ell_i}$ and is thus a sub-interval of the unit interval.

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- Also $y_1y_2y_3\ldots y_{\ell_i}$ is not a prefix of any other codeword, so the intervals must be disjoint.
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- Length of interval for codeword $y_1y_2y_3\ldots y_{\ell_i}$ is $D^{-\ell_i}$. 

Also since all intervals live in $[0,1)$ we must have

$$\sum \ell_i \frac{1}{D_{\ell_i}} \leq 1 \quad (9.5)$$

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Towards Optimal Codes

- Summarizing: Prefix code ⇔ Kraft inequality.
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- This is a constrained optimization problem:

\[
\begin{align*}
\text{minimize} & \quad \sum_i p_i \ell_i \\
\text{subject to} & \quad \sum_i D^{-\ell_i} \leq 1
\end{align*}
\] (9.7)
Towards Optimal Codes

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\end{align*}$$  \hspace{1cm} (9.7)

- Integer program is an NP-complete optimization, not likely to be efficiently solvable (unless P=NP).
Towards Optimal Codes

- Relax the integer constraints on $\ell_i$ for now, and consider Lagrangian

\[
J = \sum_i p_i \ell_i + \lambda \left( \sum_i D^{-\ell_i} - 1 \right)
\]  \hspace{1cm} (9.8)
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- Taking derivatives and setting to 0,

$$\frac{\partial J}{\partial \ell_i}$$  \hspace{1cm} (9.10)

$$\Rightarrow D^{-\ell_i} = p_i \lambda \ln D$$  \hspace{1cm} (9.11)

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(9.10)

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$$\Rightarrow D^{-\ell_i} = p_i \quad \text{yielding} \quad \ell^*_i = -\log_D p_i \quad (9.12)$$
Towards Optimal Codes

- This implies that:

\[ L^* \]

(9.13)
Towards Optimal Codes

- This implies that:

\[ L^* = \sum_i p_i \ell_i^* \]  

(9.13)

\[ L^* = \sum p_i \ell_i \]

So the optimal expected code length, as a result of this optimization process, is the entropy assuming that we are allowed to have fractional code lengths. Since \( \ell_i^* = -\log D p_i \), this means that optimal code "length" (while fractional) is the same as the information about the event. I.e., shortest possible coding length is the inherent information about an event. This is like the MDL (minimum description principle), tries to find the simplest explanation about a source.

Compare fractional codeword lengths to long block codes, what is the relation?
Towards Optimal Codes

This implies that:

\[ L^* = \sum_i p_i \ell_i^* = -\sum_i p_i \log D p_i \]  

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- This implies that:

\[ L^* = \sum_{i} p_i \ell_i^* = - \sum_{i} p_i \log_D p_i = H_D(X) \]  

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Towards Optimal Codes

This implies that:

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- Compare fractional codeword lengths to long block codes, what is the relation?
Theorem 9.3.3

Entropy is the minimum expected length. That is, the expected length $L$ of any instantaneous $D$-ary code (which thus satisfies Kraft inequality) for a r.v. $X$ is such that

$$L \geq H_D(X) \quad (9.14)$$

with equality iff $D^{-\ell_i} = p_i$. 
Proof of Theorem 9.3.3.

\[ L - H_D(X) \] (9.15)

\[ (9.17) \]

\[ (9.18) \]

\[ (9.20) \]

...
Proof of Theorem 9.3.3.

\[ L - H_D(X) = \sum_i p_i \ell_i - \sum_i p_i \log_D 1/p_i \quad (9.15) \]

\[ \geq 0 \quad \text{since} \quad c \leq 1 \quad \text{by Kraft, where} \quad c = \sum_i D - \ell_i \quad (9.20) \]
Proof of Theorem 9.3.3.

\[ L - H_D(X) = \sum_i p_i \ell_i - \sum_i p_i \log_D 1/p_i \]  \hspace{1cm} (9.15)

\[ = - \sum_i p_i \log_D D^{-\ell_i} + \sum_i p_i \log_D p_i \]  \hspace{1cm} (9.16)

\[ = \sum_i p_i \log_D (p_i \cdot r_i) - \log_D D (\sum_i D - \ell_i) \]  \hspace{1cm} (9.17)

\[ = D (p_i \| r_i) + \log_D D (1/c) \]  \hspace{1cm} (9.18)

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\[ \text{...} \]
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\[ + \sum_i p_i \log_D p_i \quad \text{(now define } r_i = \frac{D^{-\ell_i}}{\sum_i D^{-\ell_i}}) \quad (9.18) \]

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Optimal Code Lengths

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\[ = \sum_i p_i \log \frac{p_i}{r_i} - \log_D \left( \sum_i D^{-\ell_i} \right) \quad (9.20) \]

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\[ = \sum_i p_i \log \frac{p_i}{r_i} - \log_D \left( \sum_i D^{-\ell_i} \right) = D(p||r) + \log_D(1/c) \]  
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\[ = \sum_i p_i \log \frac{p_i}{r_i} - \log_D(\sum_i D^{-\ell_i}) = D(p||r) + \log_D(1/c) \]  \hspace{1cm} (9.19)

\[ \geq 0 \]  \hspace{1cm} since \( c \leq 1 \) by Kraft, where \( c = \sum_i D^{-\ell_i} \) \hspace{1cm} (9.20)

...
Optimal Code Lengths

...Proof of Theorem 9.3.3.

- So we have that \( L \geq H_D(X) \).
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\textbf{Optimal Code Lengths}

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\textbf{Definition 9.3.4 ($D$-adic)}

A probability distribution is called $D$-adic w.r.t. $D$ if each of the probabilities is $= D^{-n}$ for some $n$. 
Optimal Code Lengths

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Definition 9.3.4 ($D$-adic)

A probability distribution is called $D$-adic w.r.t. $D$ if each of the probabilities is $= D^{-n}$ for some $n$.

- Ex: when $D = 2$, the distribution $[\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}] = [2^{-1}, 2^{-2}, 2^{-3}, 2^{-3}]$ is 2-adic.
Optimal Code Lengths

...Proof of Theorem 9.3.3.

- So we have that \( L \geq H_D(X) \).
- Equality, \( L = H \) is achieved iff \( p_i = D^{-\ell_i} \) for all \( i \iff -\log_D p_i \) is an integer ...
- ...in which case \( c = \sum_i D^{-\ell_i} = 1 \)

Definition 9.3.4 (\( D \)-adic)

A probability distribution is called \( D \)-adic w.r.t. \( D \) if each of the probabilities is \( = D^{-n} \) for some \( n \).

- Ex: when \( D = 2 \), the distribution \([\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}] = [2^{-1}, 2^{-2}, 2^{-3}, 2^{-3}] \) is 2-adic.
- Thus, we have equality above iff the distribution is appropriately \( D \)-adic.
Summary of recent results

To summarize the conditions and relations:

- In the relaxed optimization problem, we relaxed the lengths so that they need not be integers, and we get $E\ell = H$.  

\[ E\ell \geq H \]
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To summarize the conditions and relations:

- In the relaxed optimization problem, we relaxed the lengths so that they need not be integers, and we get $E\ell = H$.
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- I.e., if we assume Kraft, and $\ell_i = -\log_D p_i$ is an integer, then $E\ell = H$.
- I.e., if we assume Kraft, and $\ell_i \neq -\log_D p_i$, but the lengths $\ell_i$ are still integers, then we have $E\ell > H$ strictly.
Shannon Codes

- \( L - H = D(p||r) + \log_D 1/c, \) with \( c = \sum_i D^{-\ell_i} \)
Shannon Codes

- \[ L - H = D(p||r) + \log_D 1/c, \text{ with } c = \sum_i D^{-\ell_i} \]
- Thus, to produce a code, we find closest (in the KL sense) \( D \)-adic distribution w.r.t. \( D \) to \( p \) and then construct the code as in the proof of the Kraft inequality converse.
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Shannon Codes

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This means Kraft inequality holds for these lengths, so there is a prefix code (if the lengths were too short there might be a problem but we’re rounding up).

Also, we have a bound on lengths in terms of real numbers

\[
\log_D \frac{1}{p_i} \leq \ell_i < \log_D \frac{1}{p_i} + 1 \quad (9.21)
\]
Shannon Codes

- Taking expected values on both sides yields

\[ H_D(X) \leq L < H_D(X) + 1 \]  \hspace{1cm} (9.22)
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Taking expected values on both sides yields

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Close to the entropy, only one extra bit! Is this good?
Shannon Codes

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- Also, \( H \leq L^* \leq L \) where \( L^* \) is the optimal length for integer length codes (i.e., might not have optimal integer lengths satisfying Kraft).
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**Theorem 9.4.1**

Let \( \ell_1^*, \ell_2^*, \ldots, \ell_m^* \) be the optimal integral codeword lengths for source \( p \) and \( D \)-ary alphabet. \( L^* \) is the expected length. Then

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\[ H_D(X) \leq L^* < H_D(X) + 1 \]  
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- So average overhead of using integers (rather than fractional) codeword lengths is no more than one bit per symbol.
How bad is one bit?

- How bad is this overhead?

Formula:
\[ \text{Efficiency} \triangleq H(X) - E_\ell(X) \leq 1 \] (9.24)

If \( E_\ell(X) = H(X) + 1 \), then efficiency \( \rightarrow 1 \) as \( H(X) \rightarrow \infty \).

Efficiency \( \rightarrow 0 \) as \( H(X) \rightarrow 0 \), so entropy would need to be very large for this to be good.

For small alphabets (or low-entropy distributions, such as close to deterministic distributions), impossible to have good efficiency.

E.g., \( D = \{0, 1\} \) then \( \max H(X) = 1 \), so best possible efficiency is 50%.

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- How bad is this overhead?
- Depends on $H$. Efficiency of code

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0 \leq \text{Efficiency} \triangleq \frac{H_D(X)}{E\ell(X)} \leq 1
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Improving efficiency

- Such symbol codes are inherently disadvantaged, unless their distributions are $D$-adic.
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- We can reduce overhead (improve efficiency) by coding $> 1$ symbol at a time (block code, or a vector code, the symbol is the vector).
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L_n = \frac{1}{n} \sum_{x_{1:n}} p(x_{1:n}) \ell(x_{1:n}) = \frac{1}{n} E \ell(x_{1:n})
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- Let's use Shannon coding lengths to get

$$\log \frac{1}{p_i} \leq \ell_i \leq \log \frac{1}{p_i} + 1$$  \hspace{1cm} (9.26)
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\[
\sum_{i} p_i \left( \log \frac{1}{p_i} \leq \ell_i \leq \log \frac{1}{p_i} + 1 \right)
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(9.26)

\[\Rightarrow H(X_1, \ldots, X_n) \leq E\ell(X_{1:n}) < H(X_1, \ldots, X_n) + 1\]

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Improving efficiency

- If the $X_i$ are i.i.d. then $H(X_1, \ldots, X_n) = nH(X_i)$. 
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\]

- As $n$ gets big, per symbol penalty of a Shannon code decreases, and we approach the Entropy limit (per symbol), although once again we have to code a block at a time.

- Again, even if symbols are independent it is better to code jointly.
Stochastic processes

Consider any stationary (ergodic) stochastic process. Then

\[ H(X_1, \ldots, X_n) \leq E\ell(X_1, \ldots, X_n) < H(X_1, \ldots, X_n) + 1 \]  \hspace{1cm} (9.29)

\[ nL_n < H(X_1, \ldots, X_n) + 1 \]  \hspace{1cm} (9.30)
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- Thus, as \( n \) gets large, expected length of code goes to the entropy rate of the stochastic process.
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If stationary, than l.h.s. \( \to H(X) \) as \( n \to \infty \).

Thus, as \( n \) gets large, expected length of code goes to the entropy rate of the stochastic process.

We can make penalty per source symbol as small as we want if we don't mind long block lengths. This can be stated as a theorem.
Stochastic processes

- Consider any stationary (ergodic) stochastic process. Then

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- If stationary, then l.h.s. \( \rightarrow H(\mathcal{X}) \) as \( n \rightarrow \infty \).
- Thus, as \( n \) gets large, expected length of code goes to the entropy rate of the stochastic process.
- We can make penalty per source symbol as small as we want if we don’t mind long block lengths. This can be stated as a theorem

**Theorem 9.4.2**

*Minimum expected codeword lengths per symbol satisfy*

\[ \frac{H(X_1, \ldots, X_n)}{n} \leq L^*_n < \frac{H(X_1, \ldots, X_n)}{n} + \frac{1}{n} \quad (9.31) \]

if \( X_i \) is stationary. I.e., \( L^* \rightarrow H(\mathcal{X}) \)
Coding with the wrong distribution

- In general, we don’t have the “true” distribution (if there is one).
Coding with the wrong distribution

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- In general, we don’t have the “true” distribution (if there is one).
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\[ (9.35) \]
Coding with the wrong distribution

In general, we don’t have the “true” distribution (if there is one).

With the wrong distribution, we’ll make mistakes. I.e., Shannon code would use lengths $\ell(x) = \lceil \log 1/q(x) \rceil$ but the true probability is $p(x) \neq q(x)$. How does this hurt us?

(9.35)
Coding with the wrong distribution

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$$E\ell(X)$$

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$$E\ell(X) = \sum_x p(x) \lceil \log 1/q(x) \rceil$$

(9.35)

Thus, $D(p||q)$ is per symbol bit penalty for using wrong distribution.
Coding with the wrong distribution

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\[
E\ell(X) = \sum_x p(x) \lceil \log \frac{1}{q(x)} \rceil \leq \sum_x p(x) \left( \log \frac{1}{q(x)} + 1 \right)
\]  

(9.32)
Coding with the wrong distribution

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E\ell(X) = \sum_x p(x) \lceil \log \frac{1}{q(x)} \rceil \leq \sum_x p(x) (\log \frac{1}{q(x)} + 1) \tag{9.32}
\]

\[
= \sum_x p(x) (\log \frac{p(x)}{q(x)} \frac{1}{p(x)} + 1) \tag{9.33}
\]

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Coding with the wrong distribution

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\[
Thus, D(p||q) + H(p) + 1 \quad (9.35)
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$$= D(p \| q) + H(p) + 1$$

- Thus, $D(p \| q)$ is per symbol bit penalty for using wrong distribution.
Theorem 9.4.3

Expected length under $p(x)$ of code with $\ell(x) = \lceil \log 1/q(x) \rceil$ satisfies

$$H(p) + D(p||q) \leq E_p \ell(X) \leq H(p) + D(p||q) + 1 \quad (9.36)$$

- l.h.s. is the best we can do with the wrong distribution $q$ when the true distribution is $p$. 

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Goal is to find a code with the shortest possible expected length.

From the above code class, we might think that we want to use codes from the largest class possible (since we might think we’re more likely to get shorter codes).
Kraft revisited

- We proved Kraft inequality is true for instantaneous codes (and vice versa).
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**Theorem 9.5.1**

Codeword lengths of *any* uniquely decodable code (not nec. instantaneous) must satisfy Kraft inequality \( \sum_i D^{-\ell_i} \leq 1 \).
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Theorem 9.5.1

Codeword lengths of any uniquely decodable code (not nec. instantaneous) must satisfy Kraft inequality \( \sum_i D^{-\ell_i} \leq 1 \). Conversely, given a set of codeword lengths that satisfy Kraft, it is possible to construct a uniquely decodable code.
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Proof.

Proof converse we already saw before (given lengths, we can construct a prefix code which is thus uniquely decodable). Thus we only need prove the first part.
Proof of Theorem 9.5.1.

- Given: uniquely decodable (not necessarily instantaneous) code with lengths \( \ell(x) \),
Proof of Theorem 9.5.1.

Given: uniquely decodable (not necessarily instantaneous) code with lengths \( \ell(x) \), and length of \( k \)-extension \( \ell(x, \ldots, x_k) = \sum_{i=1}^{k} \ell(x_i) \)
Kraft and uniquely decodable

Proof of Theorem 9.5.1.

- Given: uniquely decodable (not necessarily instantaneous) code with lengths $\ell(x)$, and length of $k$-extension $\ell(x, \ldots, x_k) = \sum_{i=1}^{k} \ell(x_i)$

we wish to prove that $\sum_x D^{-\ell(x)} \leq 1$. 

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- Define \( S = \sum_{x \in X} D^{-\ell(x)} \),

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$$S^k = \left[ \sum_x D^{-\ell(x)} \right]^k \quad (9.39)$$
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\[
= \sum_{x_1:k \in \mathcal{X}^k} D^{-[\sum_{i=1}^{k} \ell(x_i)]} = \sum_{x_1:k \in \mathcal{X}^k} D^{-\ell(x_1:k)} \tag{9.38}
\]

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Kraft and uniquely decodable

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\[
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Kraft and uniquely decodable

Proof of Theorem 9.5.1.

\[ k \ell_{\text{max}} \sum_{m=1}^{\infty} a(m) D^{-m} \quad (9.39) \]

- where \( \ell_{\text{max}} = \max_x \ell(x) \) is the maximum codeword length.
Proof of Theorem 9.5.1.

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- where \( \ell_{\max} = \max_x \ell(x) \) is the maximum codeword length.

- \( a(m) \) = number of source sequences \( x_{1:k} \) mapped into code words of length \( m \), i.e.,

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a(m) = \left| \left\{ x_{1:k} \in X^k : \ell(x_{1:k}) = m \right\} \right|
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Proof of Theorem 9.5.1.

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- There are \( D^m \) codewords of length \( m \), and each of them can have (at most) one associated source sequence (since code is uniquely decodable). Hence, \( a(m) \leq D^m \).
Kraft and uniquely decodable

... proof of Theorem 9.5.1.

- So continuing,

\[ S^k \]

\[ \text{(9.41)} \]
proof of Theorem 9.5.1.

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- So, \( S^k \) (exponential in \( k \)) never greater than \( k\ell_{\text{max}} \) (polynomial in \( k \)) \( \Rightarrow S \leq 1 \).
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- So, \( S^k \) (exponential in \( k \)) never greater than \( k \ell_{\max} \) (polynomial in \( k \)) \( \Rightarrow S \leq 1 \).

- Giving \( S = \sum_{x \in \mathcal{X}} D^{-\ell(x)} \leq 1 \).
Summary: uniquely decodable vs. instantaneous codes

- Set of achievable codeword lengths the same for uniquely decodable codes and for instantaneous codes.
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- So, for distortionless symbol codes, we can then just consider instantaneous codes with impunity.
- Soon, we’ll talk about stream codes where we can get the benefit of long block lengths but we don’t have to wait for the end of a block before we start decoding, which is very useful for “streaming” applications like streaming audio/video.
**Shannon Code optimal?**

- **Ex:** $\mathcal{X} = \{0, 1\}$ with $p(X = 0) = 10^{-1000} = 1 - p(X = 1)$.
Shannon Code optimal?

- Ex: $\mathcal{X} = \{0, 1\}$ with $p(X = 0) = 10^{-1000} = 1 - p(X = 1)$.
- What are Shannon lengths?

- Shannon length for symbol 0:
  \[
  \ell(0) = \lceil \log_2 \frac{1}{10^{-1000}} \rceil = 3322 \text{ bits}
  \]
- Shannon length for symbol 1:
  \[
  \ell(1) = \lceil \log_2 \frac{1}{1 - 10^{-1000}} \rceil = 1 \text{ bit}
  \]

For symbol 0, we're using 3321 too many bits.
In general, for other distributions, one can construct cases where $\lceil \log D p_i \rceil$ is longer than necessary. Shannon length codes are not optimal integer length prefix codes.
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- We wish to determine an object from this class asking as few questions as possible.
- Supposing $X \in S$, each question can take the form “Is $X \in A$?” for some $A \subseteq S$. 
Question tree. \( S = \{x_1, x_2, x_3, x_4, x_5\} \).

- \( X \in \{x_2, x_3\} \):
  - Y: \( x_2 = 0.2 \)
  - N: \( x_3 = 0.2 \)

- \( X \in \{x_1\} \):
  - Y: \( x_1 = 0.3 \)
  - N: \( X \in \{x_4\} \):
    - Y: \( x_4 = 0.15 \)
    - N: \( x_5 = 0.15 \)
20 Questions

- Question tree. \( S = \{x_1, x_2, x_3, x_4, x_5\} \).

\[
\begin{array}{c}
Y \\
X \in \{x_2\} \\
Y \\
X \in \{x_1\} \\
N \\
X \in \{x_4\} \\
Y \\
x_2 \ 0.2 \\
Y \\
x_1 \ 0.3 \\
N \\
x_4 \ 0.15 \\
N \\
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\end{array}
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- How do we construct such a tree?
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```
  Y
 / \
Y X ∈ \{x_2\} Y x_2 0.2
 / \
X ∈ \{x_2, x_3\} N x_3 0.2
 / \
N X ∈ \{x_1\} Y x_1 0.3
 / \
N X ∈ \{x_4\} Y x_4 0.15
 / \
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```

- **How do we construct such a tree?** Charles Sanders Peirce, 1901 said:

  *Thus twenty skillful hypotheses will ascertain what two hundred thousand stupid ones might fail to do. The secret of the business lies in the caution which breaks a hypothesis up into its smallest logical components, and only risks one of them at a time.*
The Greedy Method

- Suggests a greedy method. "Do next whatever currently looks best."
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The question that looks best would infer the most about the distribution, one with the largest entropy. $H(X | Y_1) = H(X, Y_1) - H(Y_1)$, so choosing a question $Y_1$ with large entropy leads to least “residual” uncertainty $H(X | Y_1)$ about $X$. Identically, we choose the question $Y_1$ with the greatest mutual information about $X$ since in this case $I(Y_1; X) = H(X) - H(X | Y_1) = H(Y_1)$.

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  \[ \{a, b, c, d, e, f, g\} = \{a, b, c, d\} \cup \{e, f, g\} \], the question “Is \( X \in \{e, f, g\}\)?” would have maximum entropy since
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- The next question depends on the outcome of the first, and we have either \(Y_1 = 0\) (\(\equiv X \in \{a, b, c, d\}\)) or \(Y_1 = 1\) (\(\equiv X \in \{e, f, g\}\)).
The Greedy Tree

- If $Y_1 = 0$ then we can split to maximize entropy as follows: partition \{a, b, c, d\} = \{a, b\} ∪ \{c, d\} since $p(\{a, b\}) = p(\{c, d\}) = 1/4$. 
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<td>({e, f}, {g})</td>
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<tr>
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\[ \{a, b, c, d\} = \{a, b\} \cup \{c, d\} \text{ since } p(\{a, b\}) = p(\{c, d\}) = \frac{1}{4}. \]

- This question corresponds to random variable $Y_2 = 1_{\{X \in \{c, d\}}$ so

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- Also, $H(X|Y_2, Y_1) = H(X, Y_2|Y_1) - H(Y_2|Y_1) = H(X|Y_1) - H(Y_2|Y_1) = H(X) - H(Y_2|Y_1) - H(Y_1)$.
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- Once we get to sets of size 2, we only have one possible question. Greedy strategy always greedily chooses what currently looks best, ignoring future. Latter questions must live with what is available.
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\[
H(Y_1, Y_2, Y_3) = H(Y_1) + H(Y_2|Y_1) + H(Y_3|Y_1, Y_2)
\]

\[= H(Y_1) + \sum_{i \in \{0,1\}} H(Y_2|Y_1 = i)p(Y_1 = i)\]  \hspace{1cm} (9.42)

\[+ \sum_{i,j \in \{0,1\}} H(Y_3|Y_1 = i, Y_2 = j)p(Y_1 = i, Y_2 = j)\]  \hspace{1cm} (9.43)
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The expected length of this code $E\ell = 2.5300$.

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Code efficiency $H/E\ell = 1.9323/2.5300 = 0.7638$.

Can we do better?
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<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>0.01</td>
<td>0.24</td>
<td>0.05</td>
<td>0.20</td>
<td>0.47</td>
<td>0.01</td>
<td>0.02</td>
</tr>
</tbody>
</table>

- This leads to the following (top-down greedily constructed) tree:

```
{a, b, c, d, e, f, g}
X ∈ {a, b, c, d}?
  X ∈ {c, d}?
    X ∈ {a, b}?
      X ∈ {e, f, g}?
        X ∈ {e}?
          X ∈ {f, g}?
            0
             0.03
              0.01
                0.02
```

- The expected length of this code $E\ell = 2.5300$.
- Entropy: $H = 1.9323$.  

#### Code Efficiency

$$\frac{H}{\ell} = \frac{1.9323}{2.5300} = 0.7638$$

Can we do better?
This leads to the following (top-down greedily constructed) tree:

\[
\{a, b, c, d, e, f, g\}
\]

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- Entropy: \( H = 1.9323 \).
- Code efficiency \( H/E\ell = 1.9323/2.5300 = 0.7638 \).
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The Greedy Tree vs. Huffman Tree

- Left is greedy, right is Huffman
The Greedy Tree vs. Huffman Tree

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![Diagram of Greedy and Huffman Trees]

The Huffman lengths have $E_{\text{huffman}} = 1.9700$.

Efficiency of Huffman code:

$$\frac{H}{E_{\text{huffman}}} = 1.9323 / 1.9700 = 0.9809$$

Key problem: Greedy procedure is not optimal in this case.
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- Key problem: Greedy procedure is not optimal in this case.
Why does starting from the top and splitting as such non-optimal? Where can it go wrong?
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Ex: There may be many ways to get a \( \approx 50\% \) split (to achieve high entropy) once done, the split is irrevocable and there is no way to know if the consequences of that split might hurt down the line.
The Huffman code tree procedure
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1. take the two least probable symbols in the alphabet.
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2. These two will be given the longest codewords, will have equal length, and will differ in the last digit.
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3. Combine these two symbols into a joint symbol having probability equal to the sum, add the joint symbol and then remove the two symbols, and repeat.
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Note that it is bottom up (agglomerative clustering) rather than top down (greedy splitting).
Huffman

- Ex: $\mathcal{X} = \{1, 2, 3, 4, 5\}$ with probabilities $\{1/4, 1/4, 1/5, 3/20, 3/20\}$. 

---

Prof. Jeff Bilmes
Huffman

- Ex: $\mathcal{X} = \{1, 2, 3, 4, 5\}$ with probabilities $\{1/4, 1/4, 1/5, 3/20, 3/20\}$.
- So 4 and 5 should have longest code length
Huffman

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<table>
<thead>
<tr>
<th>( \mathcal{X} )</th>
<th>prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.25</td>
</tr>
<tr>
<td>2</td>
<td>0.25</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
</tr>
<tr>
<td>4</td>
<td>0.15</td>
</tr>
<tr>
<td>5</td>
<td>0.15</td>
</tr>
</tbody>
</table>
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<table>
<thead>
<tr>
<th>( \mathcal{X} )</th>
<th>prob</th>
<th>step 1</th>
<th>prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.25</td>
<td>—</td>
<td>0.25</td>
</tr>
<tr>
<td>2</td>
<td>0.25</td>
<td>—</td>
<td>0.25</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>—</td>
<td>0.2</td>
</tr>
<tr>
<td>4</td>
<td>0.15</td>
<td>( 0 )</td>
<td>0.3</td>
</tr>
<tr>
<td>5</td>
<td>0.15</td>
<td>( 1 )</td>
<td>0.3</td>
</tr>
</tbody>
</table>
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<table>
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<tr>
<th>$\mathcal{X}$</th>
<th>prob</th>
<th>step 1 prob</th>
<th>step 2 prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>2</td>
<td>0.25</td>
<td>0.25</td>
<td>0.45</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>4</td>
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So we have $E\ell = 2.3$ bits and $H = 2.2855$ bits, as you can see this code does pretty well (close to entropy).

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<th>step 2</th>
<th>prob</th>
<th>step 3</th>
<th>prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.55</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.25</td>
<td>0.25</td>
<td>0.45</td>
<td>0.45</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>0.2</td>
<td></td>
<td>0.45</td>
<td>0.45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.15</td>
<td>0.3</td>
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<td></td>
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<table>
<thead>
<tr>
<th>( \log \frac{1}{p(x)} )</th>
<th>length</th>
<th>codeword</th>
<th>( \mathcal{X} )</th>
<th>prob</th>
<th>step 1</th>
<th>prob</th>
<th>step 2</th>
<th>prob</th>
<th>step 3</th>
<th>prob</th>
<th>step 4</th>
<th>prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>2</td>
<td>00</td>
<td>1</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0</td>
<td>0.55</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>2</td>
<td>10</td>
<td>2</td>
<td>0.25</td>
<td>0</td>
<td>0.45</td>
<td>0.45</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.3</td>
<td>2</td>
<td>11</td>
<td>3</td>
<td>0.2</td>
<td>0.2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.7</td>
<td>3</td>
<td>010</td>
<td>4</td>
<td>0.15</td>
<td>0</td>
<td>0.3</td>
<td>0.3</td>
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<table>
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<tr>
<th>$\log \frac{1}{p(x)}$</th>
<th>length</th>
<th>codeword</th>
<th>$\mathcal{X}$</th>
<th>prob</th>
<th>step 1</th>
<th>prob</th>
<th>step 2</th>
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<th>prob</th>
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<tr>
<td>2.0</td>
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<td>00</td>
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<td>0</td>
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<td>0</td>
<td>1.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0</td>
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<td>10</td>
<td>2</td>
<td>0.25</td>
<td>0</td>
<td>0.45</td>
<td>0.45</td>
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<table>
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<tr>
<th>( \frac{1}{p(x)} \log \frac{1}{p(x)} )</th>
<th>length</th>
<th>codeword</th>
<th>( x )</th>
<th>prob</th>
<th>step 1</th>
<th>prob</th>
<th>step 2</th>
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- In general, a particular codeword for the optimal code might be longer than Shannon’s length, but of course this is not true on average.
Optimality of Huffman

- Huffman is optimal, i.e., \( \sum_i p_i \ell_i \) is minimal, for integer lengths.
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  - First show lemma that some optimal codes have certain properties (not all, but that $\exists$ optimal code with these properties).
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To show this:

1. First show lemma that some optimal codes have certain properties (not all, but that $\exists$ optimal code with these properties).
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1. First show lemma that some optimal codes have certain properties (not all, but that $\exists$ optimal code with these properties).
2. Given a code $C_m$ for $m$ symbols, that has said properties, produce new simpler code satisfying lemma and is simpler to optimize.
3. Ultimately get down to simple case of two symbols which are obvious to optimize.
Optimality of Huffman

Lemma 9.6.1

For all distributions, \( \exists \) an optimal instantaneous code (i.e., minimal expected length) simultaneously satisfying:

1. if \( p_j > p_k \) then \( l_j \leq l_k \) (i.e., the more probable symbol does not have a longer length)
2. The two longest codewords have the same length
3. Two longest codewords differ only in last bit and correspond to the two least likely symbols.

Proof.

Suppose \( C_m \) is optimal code (so \( L(C_m) \) is minimum) and choose \( j, k \) such that \( p_j > p_k \). Need to show \( \exists \) code with \( l_j \leq l_k \).

Consider \( C'_m \) with codewords \( j \) and \( k \) swapped meaning

\[
\ell'_j = \ell_k \quad \text{and} \quad \ell'_k = \ell_j \tag{9.44}
\]

which can only make the code longer, so \( L(C'_m) \geq L(C_m) \). . .
Optimality of Huffman

... proof of lemma 9.6.1.

- With this swap, since $L(C_m)$ is minimal, we have

\[
0
\]

(9.49)

...
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\[
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Thus, \( \ell_k \geq \ell_j \) when \( p_j > p_k \) and the code satisfies property 1.

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0 \leq L(C'_m) - L(C_m) = \sum_i p_i \ell'_i - \sum_i p_i \ell_i
\]

(9.45)

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Optimality of Huffman

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$$= p_j \ell'_j + p_k \ell'_k - p_j \ell_j - p_k \ell_k$$ \hspace{1cm} (9.46)

$$\geq 0$$ \hspace{1cm} (9.49)

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Optimality of Huffman

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\]

\[
\begin{array}{l}
>0 \quad \Rightarrow \geq 0
\end{array}
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Optimality of Huffman

... proof of lemma 9.6.1.

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⇒ we have reduced expected length.
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\[
\begin{array}{cc}
\text{if siblings after deletion} & \text{if not siblings after deletion} \\
\ldots & \ldots \\
\end{array}
\]

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  - if siblings after deletion
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- Property 3 (two longest codewords differ only in last bit & correspond to two least likely source symbols).
Optimality of Huffman

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- Property 3 (two longest codewords differ only in last bit & correspond to two least likely source symbols).

- Due to property 1 ($p_k < p_j \Rightarrow \ell_k \geq \ell_j$), if $p_k$ is the smallest probability, then it must have a codeword length no less than any other $j$ with $p_j > p_k$. Similarly, if $p_k$ is second least probable, then it has codeword length no less than any more probable symbol.
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- Thus, the two longest codewords have same length (prop 2) and correspond to two least likely source symbols.
Optimality of Huffman

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\[
\begin{array}{c}
\text{pm} \\
\downarrow \\
\text{pm-1} \\
\end{array} \quad \rightarrow \quad \begin{array}{c}
\text{pm-1} \\
\downarrow \\
\text{pm} \\
\end{array}
\]

...
Optimality of Huffman

... proof of lemma 9.6.1.

- This does not change the length $L = \sum_i p_i \ell_i$. 
Optimality of Huffman

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- This does not change the length $L = \sum_i p_i \ell_i$.

- Thus, if $p_1 \geq p_2 \geq \cdots \geq p_m$, there exists an optimal code with $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_{m-1} = \ell_m$ and where $C(x_{m-1})$ and $C(x_m)$ differ only in last bit.
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- This does not change the length \( L = \sum_i p_i \ell_i \).

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- So, next we’re going to demonstrate how Huffman is optimal by starting with a code and then doing a Huffman operation to produce a new code, and where the optimization of the original code is dependent on a (simpler) optimization on a shorter code.
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We’ll continue doing this until the optimal code will be apparent.
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Assume (some not necessarily optimal) code $C_m$ (on $m$ symbols) that satisfies the above properties. $C_m$ has codewords $\{\omega_i\}_{i=1}^m$. 
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Huffman turns code $C_m$ into code $C_{m-1}$ (with codewords $\{\omega'_i\}_{i=1}^{m-1}$).
Assume (some not necessarily optimal) code $C_m$ (on $m$ symbols) that satisfies the above properties. $C_m$ has codewords $\{\omega_i\}_{i=1}^m$. Huffman builds the code backwards, taking the two smallest probabilities $p_{m-1}, p_m$, giving a bit (0 or 1) to each code word, and merges passing the result back to another round of Huffman.
**Optimality of Huffman**

- Assume (some not necessarily optimal) code $C_m$ (on $m$ symbols) that satisfies the above properties. $C_m$ has codewords $\{\omega_i\}_{i=1}^m$.

- Indices $m, m - 1$ have the least probability and longest codewords.

<table>
<thead>
<tr>
<th>$C_m$</th>
<th>length</th>
<th>symb. prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>$\ell_1$</td>
<td>$p_1$</td>
</tr>
<tr>
<td>$w_2$</td>
<td>$\ell_2$</td>
<td>$p_2$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$w_{m-2}$</td>
<td>$\ell_{m-2}$</td>
<td>$p_{m-2}$</td>
</tr>
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<td>$w_{m-1}$</td>
<td>$\ell_{m-1}$</td>
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Huffman builds the code backwards, taking the two smallest probabilities $p_{m-1}, p_m$, giving a bit (0 or 1) to each code word, and merges passing the result back to another round of Huffman.
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<td>$\ell_2$</td>
<td>$p_2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$w_{m-2}$</td>
<td>$\ell_{m-2}$</td>
<td>$p_{m-2}$</td>
</tr>
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Optimality of Huffman

Huffman implicitly goes from current code $C_m$ to $C_{m-1}$ as follows:

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</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>$\omega'_1$</td>
<td>$\ell'_1$</td>
<td>$w_1 = w'_1$</td>
<td>$\ell_1 = \ell'_1$</td>
<td>$p_1$</td>
</tr>
<tr>
<td>$p_2$</td>
<td>$\omega'_2$</td>
<td>$\ell'_2$</td>
<td>$w_2 = w'_2$</td>
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</tr>
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</tr>
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<td>$p_{m-2}$</td>
</tr>
<tr>
<td>$p_{m-1} + p_m$</td>
<td>$\omega'_{m-1}$</td>
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<td>$w_{m-1} = w'_{m-1}$</td>
<td>$\ell_{m-1} = \ell'_{m-1} + 1$</td>
<td>$p_{m-1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
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<tr>
<td>$p_1$</td>
<td>$\omega_1'$</td>
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</tr>
<tr>
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<td>$\omega_{m-2}'$</td>
<td>$\ell_{m-2}'$</td>
<td>$w_{m-2} = w_{m-2}'$</td>
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<td>$p_{m-2}$</td>
</tr>
<tr>
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<td>$\omega_{m-1}'$</td>
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<td>$w_{m-1} = w_{m-1}' 0$</td>
<td>$\ell_{m-1} = \ell_{m-1}' + 1$</td>
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</tr>
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<td></td>
<td></td>
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- Again, $\omega_i$ are the $C_m$ lengths and $\omega_i'$ are the $C_{m-1}$ lengths.
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<tr>
<td>$p_1$</td>
<td>$\omega'_1$</td>
<td>$\ell'_1$</td>
<td>$w_1 = w'_1$</td>
<td>$\ell_1 = \ell'_1$</td>
<td>$p_1$</td>
</tr>
<tr>
<td>$p_2$</td>
<td>$\omega'_2$</td>
<td>$\ell'_2$</td>
<td>$w_2 = w'_2$</td>
<td>$\ell_2 = \ell'_2$</td>
<td>$p_2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$p_{m-2}$</td>
<td>$\omega'_{m-2}$</td>
<td>$\ell'_{m-2}$</td>
<td>$w_{m-2} = w'_{m-2}$</td>
<td>$\ell_{m-2} = \ell'_{m-2}$</td>
<td>$p_{m-2}$</td>
</tr>
<tr>
<td>$p_{m-1} + p_m$</td>
<td>$\omega'_{m-1}$</td>
<td>$\ell'_{m-1}$</td>
<td>$w_{m-1} = w'_{m-1}$</td>
<td>$\ell_{m-1} = \ell'_{m-1} + 1$</td>
<td>$p_{m-1}$</td>
</tr>
</tbody>
</table>

$w_m = w'_{m-1} + 1$

Again, $\omega_i$ are the $C_m$ lengths and $\omega'_i$ are the $C_{m-1}$ lengths.

Lengths are defined recursively at the time of the Huffman step. All Huffman knows is the relationship between the current lengths and codewords (at step $m$) to the next lengths and codewords (at step $m-1$). Huffman is lazy in this way.
Optimality of Huffman

- We get the following:

\[
L(C_m) = \sum_i p_i \ell_i \quad (9.50)
\]

\[
= m - 2 \sum_i p_i \ell_i' + p_m - 1 (\ell'_m - 1 + 1) + p_m (\ell'_m - 1 + 1) \quad (9.51)
\]

\[
= m - \sum_i p_i' \ell_i' + (p_m - 1 + p_m) \ell'_m - 1 + p_m - 1 + p_m \quad (9.52)
\]

\[
= L(C_m - 1) + p_m - 1 + p_m \quad (9.53)
\]

(9.54)

Reduces num. of variables we need to optimize over.
Optimality of Huffman

- We get the following:

\[ L(C_m) \]
Optimality of Huffman

- We get the following:

\[ L(C_m) = \sum_i p_i \ell_i \]  \hspace{1cm} (9.50)

\[ L(C_{m-1}) = \sum_i p'_i \ell'_i + (p_m - 1) \ell'_m + p_m \]  \hspace{1cm} (9.54)
Optimality of Huffman

We get the following:

\[ L(C_m) = \sum_i p_i \ell_i \]  \hspace{1cm} (9.50)

\[ = \sum_{i=1}^{m-2} p_i \ell'_i + p_{m-1}(\ell'_{m-1} + 1) + p_m(\ell'_{m-1} + 1) \]  \hspace{1cm} (9.51)

(9.54)
Optimality of Huffman

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\[ = \sum_{i=1}^{m-2} p_i \ell'_i + (p_{m-1} + p_m)\ell'_{m-1} + p_{m-1} + p_m \]  \hspace{1cm} (9.52)

\[ \text{doesn't involve lengths} \]  \hspace{1cm} (9.54)
Optimality of Huffman

- We get the following:

\[ L(C_m) = \sum_{i} p_i \ell_i \]  
\[ = \sum_{i=1}^{m-2} p_i \ell_i' + p_{m-1}(\ell_{m-1}' + 1) + p_m(\ell_{m-1}' + 1) \]  
\[ = \sum_{i=1}^{m-2} p_i \ell_i' + (p_{m-1} + p_m)\ell_{m-1}' + p_{m-1} + p_m \]  
\[ = \sum_{i=1}^{m-1} p_i \ell_i' + p_{m-1} + p_m \]  
\[ = \sum_{i=1}^{m-1} p_i \ell_i' + p_{m-1} + p_m \]
Optimality of Huffman

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\[
L(C_m) = \sum_{i} p_i \ell_i
\]  
(9.50)

\[
\begin{align*}
&= \sum_{i=1}^{m-2} p_i \ell'_i + p_{m-1} (\ell'_{m-1} + 1) + p_m (\ell'_{m-1} + 1) \\
&= \sum_{i=1}^{m-2} p_i \ell'_i + (p_{m-1} + p_m) \ell'_{m-1} + p_{m-1} + p_m \\
&= \sum_{i=1}^{m-1} p'_i \ell'_i + p_{m-1} + p_m \\
&= L(C_{m-1}) + \underbrace{p_{m-1} + p_m}
\end{align*}
\]  
(9.51, 9.52, 9.53, 9.54)
Optimality of Huffman

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L(C_m) = \sum_i p_i \ell_i
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(9.51)

\[
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\]  
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\[
= \sum_{i=1}^{m-1} p_i' \ell_i' + p_{m-1} + p_m
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= L(C_{m-1}) + \underbrace{p_{m-1} + p_m}_{\text{doesn't involve lengths}}
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Optimality of Huffman

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L(C_m) = \sum_i p_i \ell_i
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\]  

\[
= \sum_{i=1}^{m-1} p'_i \ell'_i + p_{m-1} + p_m
\]  

\[
= L(C_{m-1}) + \underbrace{p_{m-1} + p_m}_{\text{doesn't involve lengths}}
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- Reduces num. of variables we need to optimize over.
Optimality of Huffman

So the Huffman procedure implies that:

\[
\min_{\ell_1:m} L(C_m) = \text{const.} + \min_{\ell_1:m-1} L(C_{m-1}) = \ldots \tag{9.55}
\]

\[
= \text{const.} + \min_{\ell_1:2} L(C_2) \tag{9.56}
\]

where each min step is Huffman, and each preserves the stated properties.
So the Huffman procedure implies that:

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where each \( \min \) step is Huffman, and each preserves the stated properties.

This reduces down to a length-2 code, which is obvious to optimize (use one bit for each source symbol), and then we backtrack to construct the code.
Optimality of Huffman

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**Theorem 9.6.2**

*The Huffman coding procedure is an optimal integer code lengths code.*
Huffman coding is a symbol code, we code one symbol at a time.
Huffman Codes

- Huffman coding is a symbol code, we code one symbol at a time.
- Is Huffman optimal?
Huffman Codes

- Huffman coding is a symbol code, we code one symbol at a time.
- Is Huffman optimal? But what does optimal mean?

In general, for a symbol code, each symbol in the source alphabet must use an integer number of codeword bits. This is okay for $D$-adic distributions but could use up to one extra bit per symbol on average. Bad example: $p(0) = 1 - p(1) = 0.999$, then $-\log p(0) \approx 0$, so we should be using close to zero bits per symbol to code this, but Huffman uses 1. Thus, we need a long block to get any benefit. In practice, this means we need to store and be able to compute $p(x_1:n)$. No problem, right?
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Prof. Jeff Bilmes
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Huffman Codes

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If $|A|$ is the alphabet size, we need a table of size $|A|^n$ to store these probabilities.
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- Moreover, it is hard to estimate $p(x_{1:n})$ accurately. Given an amount of “training data” (to borrow a phrase from machine learning), it is hard to estimate this distribution. Many of the possible strings in any finite sample size will not occur (sparsity).
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- Example: how hard is it to find a short grammatically valid English phrase never before written using a web search engine? “dogs ate banks on the river” is not found as of Mon, Oct 28, 2013.
- Smoothing models are required. Similar to the language model problem in natural language processing.
Huffman has the property that

\[ H(X) \leq L(\text{Huffman}) \leq H(X) + 1 \]  

(9.57)
Huffman Codes

- Huffman has the property that

\[ H(X) \leq L(\text{Huffman}) \leq H(X) + 1 \]  \hspace{1cm} (9.57)

- Bigger block sizes help, but we get

\[ H(X_{1:n}) \leq L(\text{Block Huffman}) \leq H(X_{1:n}) + 1 \]  \hspace{1cm} (9.58)

for the block.
Huffman Codes

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\[ H(X) \leq L(\text{Huffman}) \leq H(X) + 1 \] (9.57)

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for the block.

3. If \( H(X_{1:n}) \) is small (e.g., English text) then this extra bit can be significant.
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- Bigger block sizes help, but we get

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for the block.

- If \( H(X_{1:n}) \) is small (e.g., English text) then this extra bit can be significant.

- If block gets too long, we have the estimation problem again (hard to compute \( p(x_{1:n}) \),
Huffman Codes

- Huffman has the property that

$$H(X) \leq L(\text{Huffman}) \leq H(X) + 1 \quad (9.57)$$

- Bigger block sizes help, but we get

$$H(X_{1:n}) \leq L(\text{Block Huffman}) \leq H(X_{1:n}) + 1 \quad (9.58)$$

for the block.

- If $H(X_{1:n})$ is small (e.g., English text) then this extra bit can be significant.

- If block gets too long, we have the estimation problem again (hard to compute $p(x_{1:n})$,

- also the fact that it introduces latencies (we need to encode and then wait for the end of a block before we can send any bits).